A very common way to describe the atmosphere of a planet is by its 'scale height'. This quantity represents the vertical distance above the surface at which the density or pressure if the atmosphere decreases by exactly 1/e or $(2.718)^{-1}$ times (equal to 0.368).

The scale height, usually represented by the variable **H**, depends on the strength of the planet's gravity field, the temperature of the gases in the atmosphere, and the masses of the individual atoms in the atmosphere. The equation to the left shows how all of these factors are related in a simple atmosphere model for the density P. The variables are:

$$P(z) = P_0 \, e^{-\frac{z}{H}} \quad \text{and} \quad H = \frac{kT}{mg}$$

z: Vertical altitude in meters
T: Temperature in Kelvin degrees
m: Average mass of atoms in kilograms
g: Acceleration of gravity in meters/sec^2
k: Boltzmann's Constant 1.38×10^{-23} J/deg

Problem 1 - For Earth, g = 9.81 meters/sec^2, T = 290 K. The atmosphere consists of 22% O_2 (m= $2 \times 2.67 \times 10^{-26}$ kg) and 78% N_2 (m= $2 \times 2.3 \times 10^{-26}$ kg). What is the scale height, H?

Problem 2 - Mars has an atmosphere of nearly 100% CO_2 (m=7.3×10^{-26} kg) at a temperature of about 210 Kelvins. What is the scale height H if g= 3.7 meters/sec^2?

Problem 3 - The Moon has an atmosphere that includes about 0.1% sodium (m=6.6×10^{-26} kg). If the scale height deduced from satellite observations is 120 kilometers, what is the temperature of the atmosphere if g= 1.6 meters/sec^2?

Problem 4 - At what altitude on Earth would the density of the atmosphere P(z) be only 10% what it is at sea level, P_0?

Problem 5 - Calculate the total mass of the atmosphere in a column of air, below a height h with integral calculus. At what altitude, h, on Earth is half the atmosphere below you?

Problem 1 - Answer: First we have to calculate the average atomic mass. <m> = 0.22 ($2 \times 2.67 \times 10^{-26}$ kg) + 0.78 ($2 \times 2.3 \times 10^{-26}$ kg) = 4.76×10^{-26} kg. Then,

$$H = \frac{(1.38 \times 10^{-23})(290)}{(4.76 \times 10^{-26})(9.81)} = \textbf{8,570 meters or about 8.6 kilometers.}$$

Problem 2 - Answer:

$$H = \frac{(1.38 \times 10^{-23})(210)}{(7.3 \times 10^{-26})(3.7)} = \textbf{10,700 meters or about 10.7 kilometers.}$$

Problem 3 - Answer

$$T = \frac{(6.6 \times 10^{-26})(1.6)(120000)}{(1.38 \times 10^{-23})} = \textbf{918 Kelvins.}$$

Problem 4 - Answer: $0.1 = e^{-(z/H)}$, Take ln of both sides, $\ln(0.1) = -z/H$ then $z = 2.3\,H$ so for H= 8.6 km, **z = 19.8 kilometers**.

Problem 5 - First calculate the total mass:

$$M = \int_0^\infty P(z)dz \qquad M = P_0 \int_0^\infty e^{-\frac{z}{H}} dz \qquad M = P_0 H \int_0^\infty e^{-x} dx \qquad M = P_0 H$$

Then subtract the portion above you:

$$m = \int_h^\infty P(z)dz \qquad M = P_0 \int_h^\infty e^{-\frac{z}{H}} dz \qquad M = P_0 H \int_h^\infty e^{-x} dx \qquad M = P_0 H \left[e^{-h} - 1 \right]$$

To get: $dm(h) = P_0 H e^{-h}$

So $1/2 = e^{-h/H}$ and $h = \ln(2)(8.6 \text{ km})$ and so the height is **6 kilometers!**