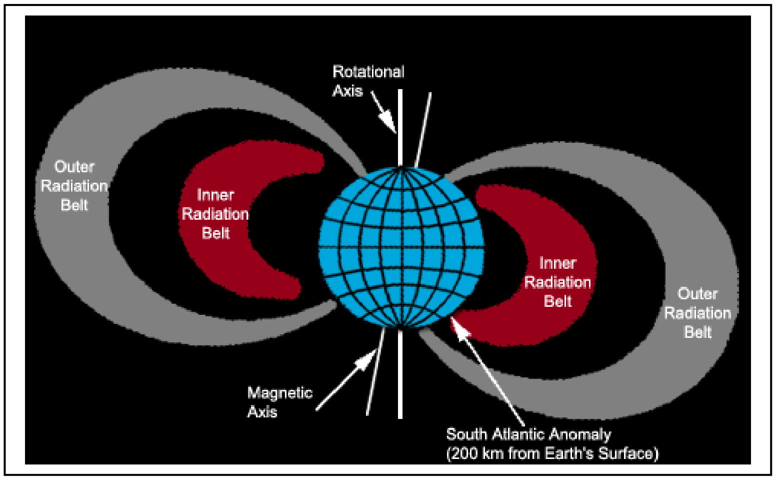
The Mass of the Van Allen Radiation Belts



The Van Allen radiation belts were discovered in the late-1950's at the dawn of the Space Age. They are high-energy particles trapped by Earth's magnetic field into donut-shaped clouds.

Earth's inner magnetic field has a 'bar magnet' shape that follows the formula

$$R(\lambda) = L \cos^2\lambda$$

where the angle, λ, is the magnetic latitude of the magnetic field line emerging from Earth's surface, and L is the distance to where that field line passes through the magnetic equatorial plane of the field. The distance, L, is conveniently expressed in multiples of Earth's radius (1 Re = 6378 kilometers) so that L=2 Re indicates a field line that intersects Earth's magnetic equatorial plane at a physical distance of 2 x 6378 km = 12,756 km from Earth's center.

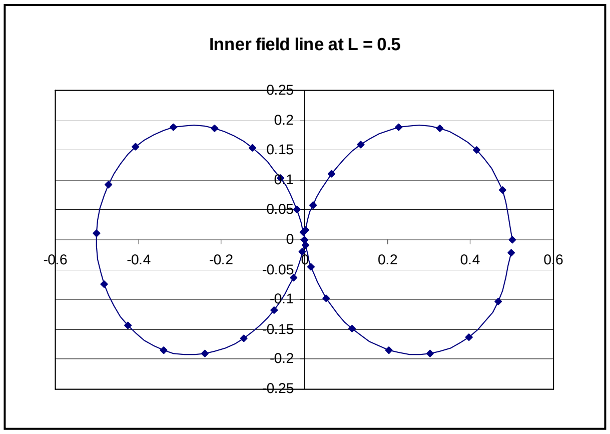
To draw a particular field line, you select L, and then plot R for different values of λ. Because the Van Allen particles follow paths along these field lines, the shape of the radiation belts is closely related to the shape of the magnetic field lines.

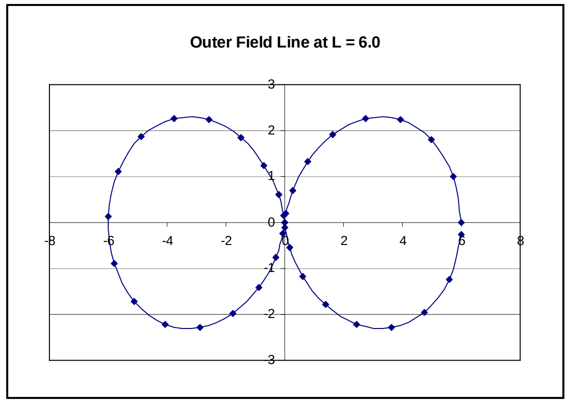
Problem 1 - Using the field line equation, plot in polar coordinates a field line at the outer boundary of the Van Allen belts for which L = 6 Re, and on the same plot, a field line at the inner boundary where L=0.5 Re. Shade-in the region bounded by these two field lines.

Problem 2 – If you rotate the shaded region in Problem 1 you get a 3-d figure which looks a lot like two nested toroids. Approximate the volume of the shaded region by using the equation for the volume of a torus given by $V = 2\pi^2 r R^2$ where R is the internal radius of the circular cross-section of the torus, and r is the distance from the Origin (Earth) to the central axis of the torus. (Think of the volume as turning the torus into a cylinder with a cross section of πR^2 and a height of $2\pi r$).

Problem 3 - Assuming that the maximum, average density of the Van Allen belts is about 10^{-5} protons/cm^3, and that the mass of a proton is 1.6×10^{-24} grams, what is the total mass of the Van Allen belts in kilograms?

Problem 1 - This may be done, either using an HP-83 graphing calculator, or an Excel spreadsheet. The later example is shown below. (Note the scale change). For Cartesian plots in Excel, (X-Y) you will need to compute X and Y parametrically as follows: (Polar to Cartesian coordinates) $X = R\cos(\lambda)$, $y = R\sin(\lambda)$, then since $R = L\cos^2(\lambda)$ we get $X = L\cos^3(\lambda)$ and $Y = L\cos^2(\lambda)\sin(\lambda)$.





Problem 2 – Answer: The outer torus of the Van Allen belt model has an internal radius of R = (6Re – 0.5 Re)/2 = 2.75 Re or 17540 km. The radius of the belts, r = 2.75 Re or 17,540 km. This makes the total volume V(outer) = $2\,(3.141)^2\,(17540 \text{ km x}1000 \text{ m/km})^3 = 1.1 \times 10^{23}$ meters^3. The volume of the inner torus is defined by R = 0.25Re = 1595 km and r = 0.25 Re = 1595 km, so its volume is V(inner) = $2\,(3.141)^2\,(1595 \text{ km x}1000 \text{ m/km})^3 = 8.0 \times 10^{19}$ meters^3 The approximate volume of the shaded region in Problem 1 is then the difference between V(outer) and V(inner) or 1.1×10^{23} meters^3 - 8.0×10^{19} meters^3 = $11000 \times 10^{19} – 8.0 \times 10^{19}$ = **1.1×10^{23} meters^3** because although it is technically correct to subtract the inner volume (containing no belt particles) from the outer volume, practically speaking, it makes no difference numerically. This would not be the case if we had selected a much larger inner boundary zone for the problem!

Problem 3 - Assuming that the maximum, average density of the Van Allen belts is about 10^{-5} protons/cm^3, and that the mass of a proton is 1.6×10^{-24} grams, what is the total mass of the Van Allen belts in kilograms?

Answer: Mass = density x Volume, V = 1.1×10^{23} meters^3 .
D = 10^{-5} protons/cm^3 x 1.6×10^{-24} grams/proton = 1.6×10^{-29} grams/cm^3 which, when converted into MKS units gives 1.6×10^{-29} g/cm^3 x (1 kg/1000 gm) x (100 cm/1 meter)^3 = 1.6×10^{-26} kg/m^3. So the total mass is about M = 1.6×10^{-26} kg/m^3 x 1.1×10^{23} meters^3 and so **M = 0.00018 kilograms.**