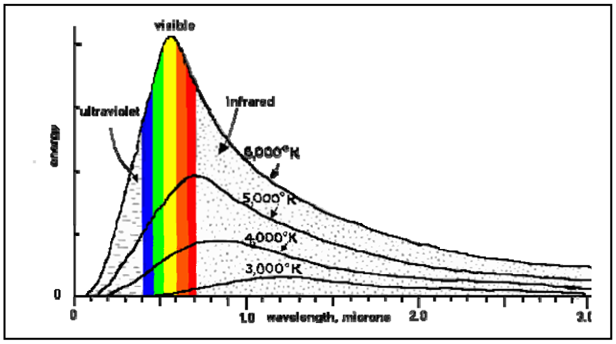
Why are hot things red?



When radiation is produced by a heated body, the intensity of electromagnetic radiation depends on frequency (wavelength) in a manner defined by the Planck Function. There is a simple law, called the Wein Displacement Law, that relates the temperature of a body to the frequency where the Planck curve has its maximum value. In this exercise, we will use two different methods to derive this law.

$$I(\lambda, T) = \frac{A}{\lambda^5 \left(e^{\left[\frac{14394}{T\lambda}\right]} - 1 \right)}$$

where: $A = 3.747 \times 10^{14}$ watts microns^4/m^2/str

Temperature (K)	Peak Wavelength (microns)
10,000	**0.2898**
9,000	**0.322**
8,000	**0.362**
7,000	**0.414**
6,000	**0.483**
5,000	**0.579**
4,000	**0.724**
3,000	**0.966**
2,000	**1.449**
1,000	**2.828**
500	**5.796**
300	**9.660**

Algebra Problem:

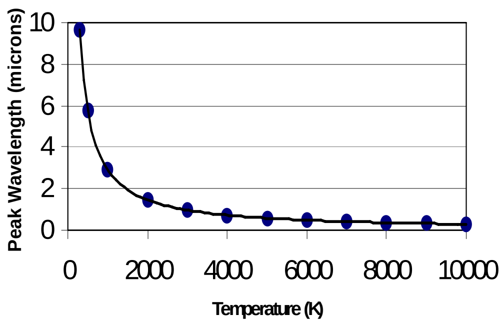
A) From the data in the table, use a calculator to find a formula that fits the data. Some possibilities might include a linear equation, $\lambda = a\,T$, power laws such as $\lambda = b\,T^{-1}$, $\lambda = c\,T^{-2}$ or $\lambda = d\,T^{2}$, $\lambda = e\,T^{3}$ or exponential functions such as $\lambda = f\,e^{(gT)}$ where a,b,c,d,e,f,g are constants determined by the fitting process. B) Which function fits the tabulated data the best, and what is the value of the constant?

Calculus Problem:

A) Find Equation 2 for the maximum of the function $I(\lambda,T)$ by differentiating with respect to the wavelength, λ, and setting the derivative equal to zero.

B) Find the solution to Equation 2, which you found in Part A, using the technique of 'successive approximation', or 'trial and error'. (Note, ignore trivial solutions involving zero! From this iterated solution, find the form of the function for the maximum wavelength as a function of temperature.)

Answer Key:



Algebra Problem: The figure above shows the data plotted in the table, and the best fit curve. This was done by copying the table into an Excel spread sheet, plotting the data as an XY scatter plot, and using 'Add Trend line'. Students may experiment with various choices for the fitting function using an HP-83 graphing calculator, or the Excel spread sheet trend line options, but should find that the best fit has the form: $\lambda = b\ T^{-1}$ where b = 2898.0 micronsKelvins. Students may puzzle over the units 'micronsKelvins' but it is often the case in physics that units have complex forms that are not immediately intuitive.

Calculus Problem: A) Below is a recommended strategy:

Let $U = A\ \lambda^{-5}$ then $dU/d\lambda = -5\ A\ \lambda^{-6}$

Let $V = e^{(14329/(\lambda\ T))} - 1$ then $dV/d\lambda = -14329 / (\lambda^2\ T)\ e^{14329/(\lambda t)}$

Then use the quotient rule: $d/d\lambda\ (U/V) = 1/V\ dU/d\lambda - U/V^2\ dV/d\lambda$

To get $dU/d\lambda - U/V\ dV/d\lambda = 0$

Then by substitution and a little algebra

$5\ l\ T\ (e^{14329/(\lambda T)} - 1) - 14329\ e^{14329/\lambda T} = 0$

Let $X = 14329/(\lambda\ T)$ then we get a simpler equation to solve:

$5(e^x - 1) - x\ e^x = 0$ (Equation 2)

Equation 3, when solved, will give the location of the extrema for the Planck Function, however, it cannot be solved exactly. You will need to program a graphing calculator or use an Excel spreadsheet to find the value for X that gives, in this case, the maximum value of the Planck Function.

Calculus Problem: B) The table shows some trial-and-error results for Equation 3, and a convergence to approximately X = 4.965 The formula for the peak wavelength is then

$4.965 = 14394 / (\lambda\ T)$ or $\mathbf{\lambda = 2899 / T}$

This is similar to the function derived by fitting the tabulated data.

X	Eq. 3
1	5.873127
2	17.16717
3	35.17107
4	49.59815
4.5	40.00857
4.9	8.428978
4.95	2.058748
4.96	0.703752
4.965	0.015799
4.966	-0.12263