As they are forming, planets and raindrops grow by accreting matter (water or asteroids) at their surface.

The basic shape of a planet or a raindrop is that of a sphere. As the sphere increases in size, there is more surface area for matter to be accreted onto it, and so the growth rate increases.

In the following series of problems, we are going to follow a step-by-step logical process that will result in a simple mathematical model for predicting how rapidly a planet or a raindrop forms.

Problem 1 - The differential equation for the growth of the mass of a body by accretion is given by Equation 1 and the mass of the body is given by Equation 2

Equation 1) $\frac{dM}{dt} = 4\pi\ \rho\ V\ R(t)^2$ Equation 2) $M(t) = \frac{4}{3}\pi\ D\ R(t)^3$

where R is the radius of the body at time t, V is the speed of the infalling material, ρ is the density of the infalling material, and D is the density of the body.

Solve Equation 2 for R(t), substitute this into Equation 1 and simplify.

Problem 2 - Integrate your answer to Problem 1 to derive the formula for M(t).

Raindrop Condensation - A typical raindrop might form so that its final mass is about 0.0001 kilograms and D = 1000 kg/m^3, under atmospheric conditions where ρ = 1 kg/m^3 and V = 1 m/sec. How long would it take such a raindrop to condense?

Planet Accretion - A typical rocky planet might form so that its final mass is about that of Earth or 5.9x10^{24} kg, and D = 3000 kg/m^3, under conditions where ρ = 0.000001 kg/m^3 and V = 1 km/sec. How long would it take such a planet to accrete using this approximate mathematical model?

Figure from 'Planetary science: Building a planet in record time' by Alan Brandon, Nature 473,460–461(26 May 2011)

Answer 1:

$$R(t) = \left(\frac{3M(t)}{4\pi D}\right)^{\frac{1}{3}} \quad \text{then} \quad \frac{dM}{dt} = 4\pi\ \mathrm{V}\rho\left(\frac{3M}{4\pi D}\right)^{\frac{2}{3}} \quad \text{so} \quad \frac{dM}{dt} = 4\pi V\rho\left(\frac{3}{4\pi D}\right)^{\frac{2}{3}}\ \mathrm{M}^{\frac{2}{3}}$$

Answer 2: First re-arrange the terms to form the integrands:

$$\frac{dM}{M^{\frac{2}{3}}} = 4\pi \mathrm{V}\rho\left(\frac{3}{4\pi D}\right)^{\frac{2}{3}} dt \qquad \text{Integrate both sides:} \quad 3M^{\frac{1}{3}} = 4\pi V\rho\left(\frac{3}{4\pi D}\right)^{\frac{2}{3}} t$$

Now solve for M(t) to get the answer: $M(t) = \left(\frac{4\pi V\rho}{3}\right)^{3}\left(\frac{3}{4\pi D}\right)^{2} t^{3}$

Raindrop Condensation - A typical raindrop might form so that its final mass is about 0.0001 kilograms and **D** = 1000 kg/m^{3}, under atmospheric conditions where ρ = 1 kg/m^{3} and **V** = 2.0 m/sec (about 5 miles per hour). How long would it take such a raindrop to condense using this approximate mathematical model?

$$0.0001 = \left(\frac{4\pi(2.0)(1.0)}{3}\right)^{3}\left(\frac{3}{4\pi(1000)}\right)^{2} t^{3} \qquad \text{so} \quad t^{3} = 2.98$$

and so it takes about **1.4 seconds**.

Planet Accretion - A typical rocky planet might form so that its final mass is about that of Earth or 5.9x10^{24} kg, and **D** = 3000 kg/m^{3}, under conditions where ρ = 0.000001 kg/m^{3} and **V** = 1 km/sec. How long would it take such a planet to accrete using this approximate mathematical model?

$$5.9x10^{24} = \left(\frac{4\pi(1000)(0.000001)}{3}\right)^{3}\left(\frac{3}{4\pi(3000)}\right)^{2} t^{3} \qquad \text{so} \quad t^{3} = 1.27\text{x}10^{40}$$

and so it takes about 2.3x10^{13} seconds or **750,000 years**.