Compound Interest

How it works: Suppose this year I put $100.00 in the bank. The bank invests this money and at the end of the year gives me $4.00 back in addition to what I gave them. I now have $104.00. My initial $100.00 increased in value by 100% x ($104.00 - $100.00)/ $100.00) = 4%. Suppose I gave all of this back to the bank and they reinvested in again. At the end of the second year they have me another 4% increase. How much money do I now have? I get back an additional 4%, but this time it is 4% of $104.00 which is $104.00 x 0.04 = $4.16. Another way to write this after the second year is:

$$\$100.00 \times (1.04) \times (1.04) = \$108.16.$$

After 6 years, at a gain of 4% each year, my original $100.00 is now worth:

$$\$100.00 \times (1.04) \times (1.04) \times (1.04) \times (1.04) \times (1.04) \times (1.04) = \$126.53$$

Do you see the pattern? The basic formula that lets you calculate this 'compound interest' easily is:

$$F = B \times (1 + P/100)^T$$

where :
B = the starting amount, P= the annual percentage increase, T = number of investment years.

Question: In the formula, why did we divide the interest percentage by 100 and then add it to 1?

Problem 1: The US Space program invested $26 billion to build the Apollo Program to send 7 missions to land on the Moon.

A) What was the average cost for each Apollo mission?

B) Since the last moon landing in 1972, inflation has averaged about 4% each year. From your answer to A), how much would it cost to do the same Apollo moon landing in 2007?

Problem 2: A NASA satellite program was originally supposed to cost $250 million when it started in 2000. Because of delays in approvals by Congress and NASA, the program didn't get started until 2005. If the inflation rate was 5% per year, A) how much more did the mission cost in 2005 because of the delays? B) Was it a good idea to delay the mission to save money in 2000?

Problem 3: A scientist began his career with a salary of $40,000 in 1980, and by 2000 this had grown to $100,000. A) What was his annual salary gain each year? B) If the annual inflation rate was 3%, why do you think that his salary gain was faster than inflation during this time?

Answer Key

Do you see the pattern? Each year you invest the money, you multiply what you started with the year before by 1.04.

$$F = B \times (1 + P/100)^{T}$$

Question: In the formula, why did we divide the interest percentage by 100 and then add it to 1? Because if each year you are increasing what you started with by 4%, you will have 4% more at the end of the year, so you have to write this as 1 + 4/100 = 1.04 to multiply it by the amount you started with.

Problem 1: The US Space program invested $26 billion to build the Apollo Program to send 7 missions to land on the Moon. A) What was the average cost for each Apollo mission?

Answer : $26 billion/7 = $3.7 billion.

B) Answer: The number of years is 2007-1972 = 35 years. Using the formula, and a calculator:

$$F = \$3.7 \text{ billion} \times (1 + 4/100)^{35} = \$3.7 \text{ billion} \times (1.04)^{35} = \$14.6 \text{ billion.}$$

Problem 2: A) Answer: The delay was 5 years, so

$$F = \$250 \text{ million} \times (1 + 5/100)^{5} = \$250 \text{ million} \times (1.28) = \$319 \text{ million}$$

The mission cost $69 million more because of the 5-year delay.

B) No, because you can't save money starting an expensive mission at a later time. Because of inflation, missions always cost more when they take longer to start, or when they take longer to finish.

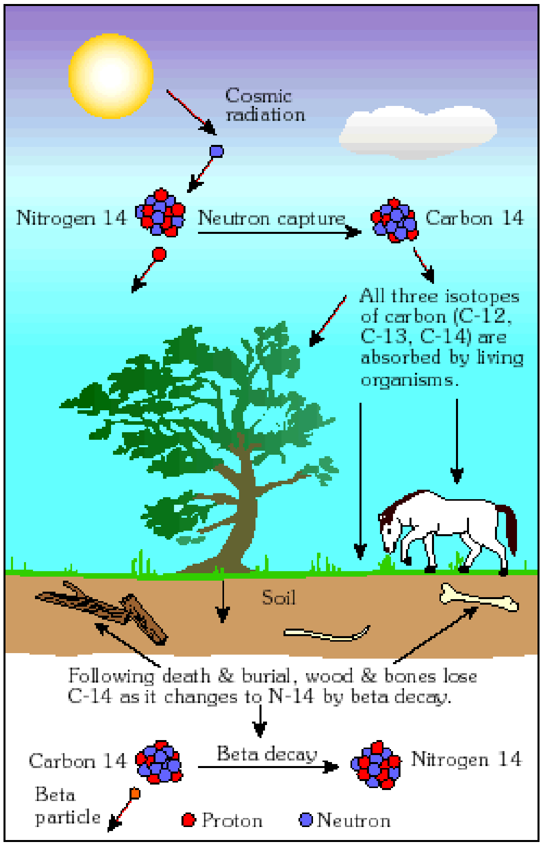
Problem 3: A scientist began his career with a salary of $40,000 in 1980, and by 2000 this had grown to $100,000. A) What was his annual salary gain each year? Answer A) The salary grew for 20 years, so using the formula and a calculator, solve for X the annual growth:

$$\$100{,}000 = \$40{,}000 \times (X)^{20} \qquad X = (100{,}000/40{,}000)^{1/20} \qquad X = 1.047$$

So his salary grew by about 4.7% each year, which is a bit faster than inflation.

B) If the inflation rate was 3%, why do you think that his salary gain was faster than inflation during this time? Answer: His salary grew faster than inflation because his employers valued his scientific research and gave him average raises of 1.5% over inflation each year!

The number 'e'



Most elements come in several varieties called isotopes, which only differ in the number of neutrons that they contain. Most isotopes are unstable, and will decay into more stable isotopes or elements over time.

The decay time is measured by the time it takes half of the atoms to change into other forms and is called the half-life. A simple formula based on powers of the number 'e' connect the initial number of atoms to the remaining number after a time period has passed:

$$N(t)=a\ e^{-0.69t/T}$$

where T is the half-life in the same time units as t.

Problem 1 – What is the initial number of atoms at a time of t=0?
Answer: N(t) = a

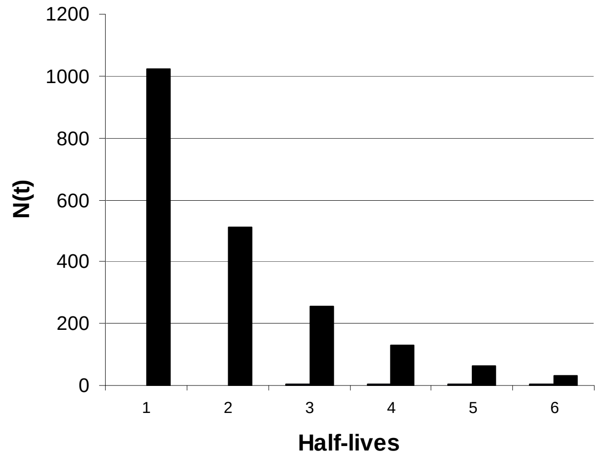
Problem 2 – Use a bar-graph to plot the function N(t) for a total of 6 half-lives with a = 2048.

Problem 3 - If a=1000 grams and T = 10 minutes, what will be the value of N(t) in when t = 1.5 hours?

Problem 4 – Carbon has an isotope called 'carbon-14' that decays to ordinary nitrogen in 5770 years. Suppose that a sample of plant material started out with 10 grams of carbon-14. If the half-life 5770 years, how many grams of carbon-14 will be present after 3,000 years?

Problem 1 – What is the initial number of atoms at a time of t=0?
Answer: **N(0) = a**

Problem 2 – Use a bar-graph to plot the function N(t) for a total of 6 half-lives with a = 2048. Answer: N = 1024, 512, 256, 128, 64, 32



Problem 3 - If a=1000 grams and T = 10 minutes, what will be the value of N(t) in when t = 1.5 hours?

Answer: 1.5 hours = 90 minutes, so since t and b are now in the same time units:

$N(90 \text{ minutes}) = 1000 \times e^{(-0.69*90/10)}$
$= 1000 \times 0.002$
= 2 grams

Problem 4 – Carbon has an isotope called 'carbon-14' that decays to ordinary nitrogen in 5770 years. Suppose that a sample of plant material started out with 10 grams of carbon-14. If the half-life is 5770 years, how many grams of carbon-14 will be present after 3,000 years?
Answer: a = 10 grams, T = 5770 years so $N(t) = 10\ e^{-0.69(t/5770)}$
After t = 3000 years,
$N(3000) = 10\ e^{-0.69(3000/5770)}$
N(3000) = 10 (0.7) **N(3000) = 7 grams.**

The number 'e'

After a star becomes a supernova, the light that its expanding gas produces fades over time. Astronomers have discovered that this fade-out is controlled by the light produced by the decay of radioactive nickel atoms.

This series of two photographs were taken by Dr. David Malin at the Anglo-Australian Observatory in 1987 and shows a before-and-after view of the supernova of 1987.

Careful studies of the brightness of this supernova in the years following the explosion reveal the 'radioactive decay' of its light.

Supernova 1987A produced 24,000 times the mass of our Earth in nickel-56 atoms, which were ejected into the surrounding space and began to decay to a stable isotope called cobalt-56. The half-life of nickel-56 is 6.4 days. Eventually the cobalt-56 atoms began to decay into stable iron-56 atoms. The half-life for the cobalt decay is 77 days.

Problem 1 – Using the half-life formula $N(t) = a\ e^{-0.69(t/T)}$, how much of the original nickel-56 (with T = 6.4 days) was still present in the supernova debris after 100 days?

Problem 2 – Assuming that no further light is produced by the nickel-56 decays after 100 days, and that for t > 100 days the light is produced by cobalt-56 decay:

A) Create a table showing the predicted brightness, L, of this supernova between 100 days and 900 days (2.5 years) after the explosion if at t=100 days the brightness of the supernova equals 80 million times that of the sun. (Answers to 2 significant figures;

B) Graph the data, called a light curve, L(t), for the first 500 days of the decay.

C) How long did it take for the supernova to fade until it exactly equaled the luminosity of our sun (L = 1.0)?

Problem 1 – Using the half-life formula $N(t) = a\,e^{\mathbf{-0.69(t/T)}}$, how much of the original nickel-56 (with T = 6.4 days) was still present in the supernova debris after 100 days?

Answer: The paragraph says that the supernova produced 24,000 times the mass of the earth in nickel-56, so a = 24,000 and for T = 6.4 days we have

$N(100 \text{ days}) = 24000\,e^{-0.69(100/6.4)}$

$N(100 \text{ days}) = 24000\,(0.00021)$

N(100 days) = **0.5 times the Earth's mass!**

Problem 2 – Assuming that no further light is produced by the nickel-56 decays after 100 days, and that for t > 100 days the light is produced by cobalt-56 decay:

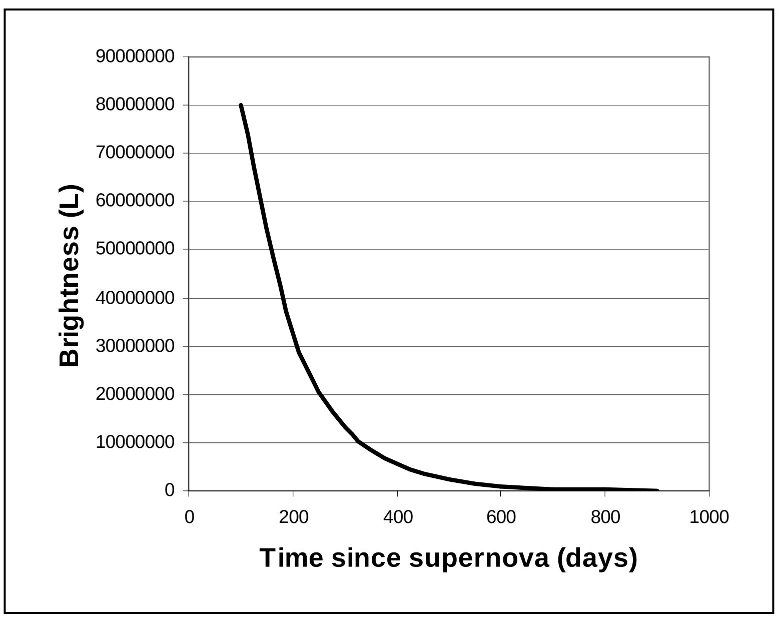
A) Create a table showing the predicted brightness, L, of this supernova between 100 days and 900 days (2.5 years) after the explosion if at t=100 days the brightness of the supernova equals 80 million times that of the sun (Answers to 2 significant figures);
Answer below.

B) Graph the data, called a light curve, L(t), for the first 500 days of the decay.
Answer below.

C) How long did it take for the supernova to fade until it exactly equaled the luminosity of our sun (L = 1.0)?
Answer: Solve $1.0 = 80 \text{ million } e^{\mathbf{(-0.69(t-100)/77)}}$.

$\ln(1.0/80 \text{ million}) = -0.69(t-100)/77$ so $t - 100 = 77 \ln(80{,}000{,}000)/0.69$ and so
t = 2,131 days or 5.8 years.

Days	L(t)
100	80,000,000
200	33,000,000
300	13,000,000
400	5,400,000
500	2,200,000
600	910,000
700	370,000
800	150,500
900	62,000



Logarithmic Functions

One of the very first things that astronomers studied was the number of stars in the sky. From this, they hoped to get a mathematical picture of the shape and extent of the entire Milky Way galaxy. This is perhaps why some cartoons of 'astronomers' often have them sitting at a telescope and tallying stars on a sheet of paper! Naked-eye counts usually number a few thousand, but with increasingly powerful telescopes, fainter stars can be seen and counted, too.

Over the decades, sophisticated 'star count' models have been created, and rendered into approximate mathematical functions. One such approximation, which gives the average number of stars in the sky, is shown below:

$$\text{Log}_{10}N(m) = -0.0003\ m^3 + 0.0019\ m^2 + 0.484\ m - 3.82$$

This polynomial is valid over the range [+4.0, +25.0] and gives the Log_{10} of the total number of stars per square degree fainter than an apparent magnitude of m. For example, at an apparent magnitude of +6.0, which is the limit of vision for most people, the function predicts that $\text{Log}_{10}N(6) = -0.912$ so that there are $10^{-0.912} = 0.12$ stars per square degree of the sky. Because the full sky area is 41,253 square degrees, there are about 5,077 stars brighter than, or equal to, this magnitude.

Problem 1 - A small telescope can detect stars as faint as magnitude +10. If the human eye-limit is +6 magnitudes, how many more stars can the telescope see than the human eye?

Problem 2 - The Hubble Space Telescope can see stars as faint as magnitude +25. About how many stars can the telescope see in an area of the sky the size of the full moon (1/4 square degree)?

Problem 1 - Answer: From the example, there are 0.12 stars per square degree brighter than +6.0

$$\text{Log}_{10}N(+10) = -0.0003\ (10)^3 + 0.0019\ (10)^2 + 0.484\ (10) - 3.82$$
$$= -0.3 + 0.19 + 4.84 - 3.82$$
$$= +0.55$$

So there are $10^{.55}$ = 3.55 stars per square degree brighter than +10. Converting this to total stars across the sky (area = 41,253 square degrees) we get 5,077 stars brighter than +6 and 146,448 stars brighter than +10. The number of additional stars that the small telescope will see is then 146,448 - 5,077 = **141,371 stars.**

Problem 2 - Answer: $\text{Log}_{10}N(25) = -0.0003\ (25)^3 + 0.0019\ (25)^2 + 0.484\ (25) - 3.82$
$= +4.78$

So the number of stars per square degree is $10^{+4.78}$ = 60,256. For an area of the sky equal to 1/4 square degree we get (60,256) x (0.25) = **15,064 stars**.

Astronomers and physicists often find linear plotting scales very cumbersome to use because the quantities you would most like to graph differ by powers of 10 in size, temperature or mass. Log-Log graphs are commonly used to see the 'big picture'. Instead of a linear scale '1 kilometer, 2 kilometers 3 kilometers etc' a Logarithmic scale is used where '1' represents 10^1, '2' represents 10^2 …'20' represents 10^{20} etc. Below we will work with a Log(T) log(D) graph where T is the temperature, in Kelvin degrees, of matter and D is its density in kg/m^3.

Problem 1 - Plot some or all of the objects listed in the table below on a Log-Log graph with the 'x' axis being Log(D) and 'y' being Log(T).

Problem 2 - A) Draw a line that includes the three black hole objects, and shade the region that forbids objects denser or cooler than this limit. B) Draw a line, and shade the region that represents the quantum temperature limit, which says that temperatures may not exceed $T<10^{32}$ K.

Problem 3 - On this graph, plot the curve representing the temperature, T, and density D, of the Big Bang at a time, t, seconds after the Big Bang given by

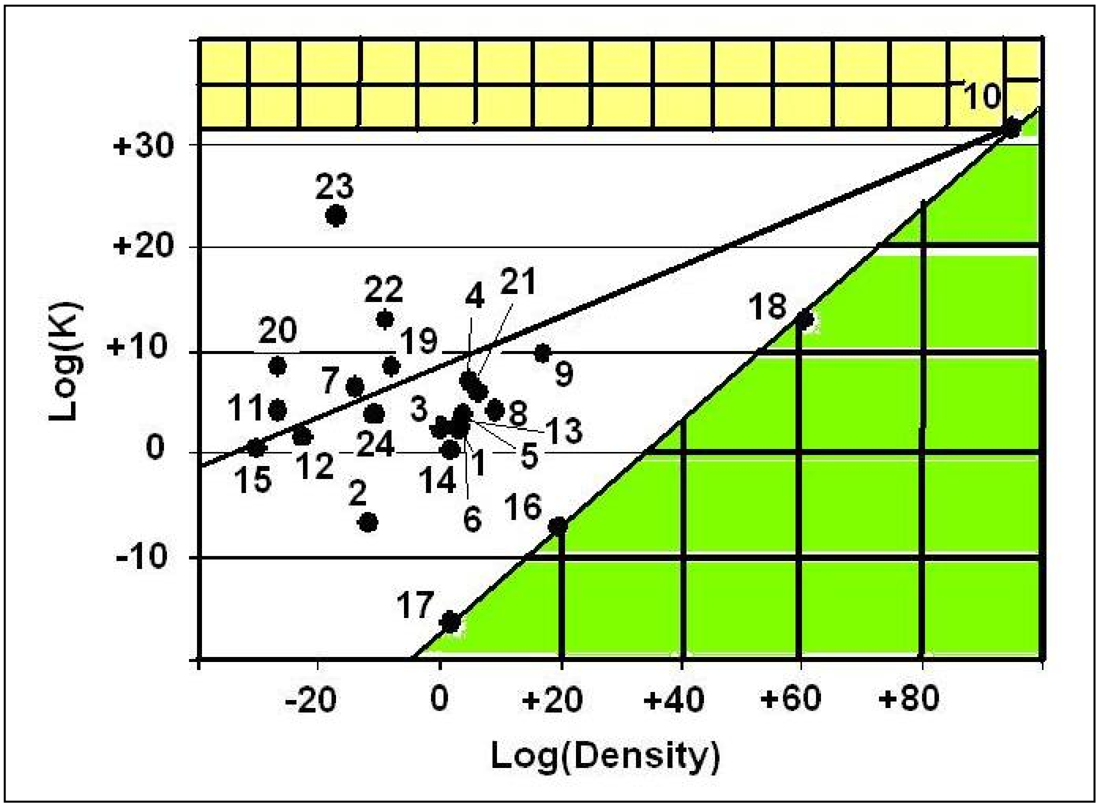
$$T = 1.5 \times 10^{10}\ t^{-\frac{1}{2}}\ \text{K} \qquad \text{and} \qquad D = 4 \times 10^{8}\ t^{-2}\ \text{kg/m}^3$$

	Object or Event	D (kg/m^3)	T (K)
1	Human	1000	290
2	Bose-Einstein Condensate	2×10^{-12}	2×10^{-7}
3	Earth atmosphere @ sea level	1.0	270
4	Core of the sun	1×10^{5}	1×10^{7}
5	Core of Earth	1×10^{4}	6×10^{3}
6	Water at Earth's surface	1×10^{3}	270
7	Solar corona	2×10^{-14}	2×10^{6}
8	White dwarf core	2×10^{9}	2×10^{4}
9	Neutron star core	2×10^{17}	4×10^{9}
10	Quantum limit	4×10^{94}	2×10^{32}
11	Interstellar medium - cold	2×10^{-27}	2×10^{4}
12	Dark interstellar cloud	2×10^{-23}	40
13	Rocks at surface of the Earth	3×10^{3}	270
14	Liquid Helium	1×10^{2}	2
15	Cosmic background radiation	5×10^{-31}	3
16	Solar-mass Black Hole	7×10^{19}	6×10^{-8}
17	Supermassive black hole	100	6×10^{-17}
18	Quantum black hole	3×10^{60}	1×10^{13}
19	Controlled fusion Tokamak Reactor	1×10^{-8}	2×10^{8}
20	Intergalactic medium - hot	2×10^{-27}	2×10^{8}
21	Brown dwarf core	2×10^{6}	1×10^{6}
22	Cosmic gamma-rays (1 GeV)	1×10^{-9}	1×10^{13}
23	Cosmic gamma-rays (10 billion GeV)	1×10^{-17}	1×10^{23}
24	Starlight in the Milky Way	2×10^{-11}	6,000

The figure below shows the various items plotted, and excluded regions cross-hatched. Students may color or shade-in the permitted region.

.

Inquiry: Can you or your students come up with more examples of objects or systems that occupy some of the seemingly 'barren' regions of the permitted area?



Logarithmic Functions

The universe is a BIG place…but it also has some very small ingredients! Astronomers and physicists often find linear plotting scales very cumbersome to use because the quantities you would most like to graph differ by powers of 10 in size, temperature or mass. Log-Log graphs are commonly used to see the 'big picture'. Instead of a linear scale '1 kilometer, 2 kilometers 3 kilometers etc' a Logarithmic scale is used where '1' represents 10^1, '2' represents 10^2 …'20' represents 10^{20} etc. A calculator easily lets you determine the Log of any decimal number. Just enter the number, n, and hit the 'log' key to get m = log(n). Then just plot a point with 'm' as the coordinate number!

Below we will work with a Log(m) log(r) graph where m is the mass of an object in kilograms, and r is its size in meters.

Problem 1 - Plot some or all of the objects listed in the table below on a LogLog graph with the 'x' axis being Log(M) and 'y' being Log(R).

Problem 2 - Draw a line that represents all objects that have a density of A) nuclear matter (4×10^{17} kg/m^3), and B) water (1000 kg/m^3).

Problem 3 - Black holes are defined by the simple formula R = 3.0 M, where r is the radius in kilometers, and M is the mass in multiples of the sun's mass (1 M = 2.0×10^{30} kilograms). Shade-in the region of the LogLog plot that represents the condition that no object of a given mass may have a radius <u>smaller</u> than that of a black hole.

Problem 4 - The lowest density achievable in our universe is set by the density of the cosmic fireball radiation field of 4×10^{-31} kg/m^3. Draw a line that identifies the locus of objects with this density, and shade the region that excludes densities <u>lower</u> than this.

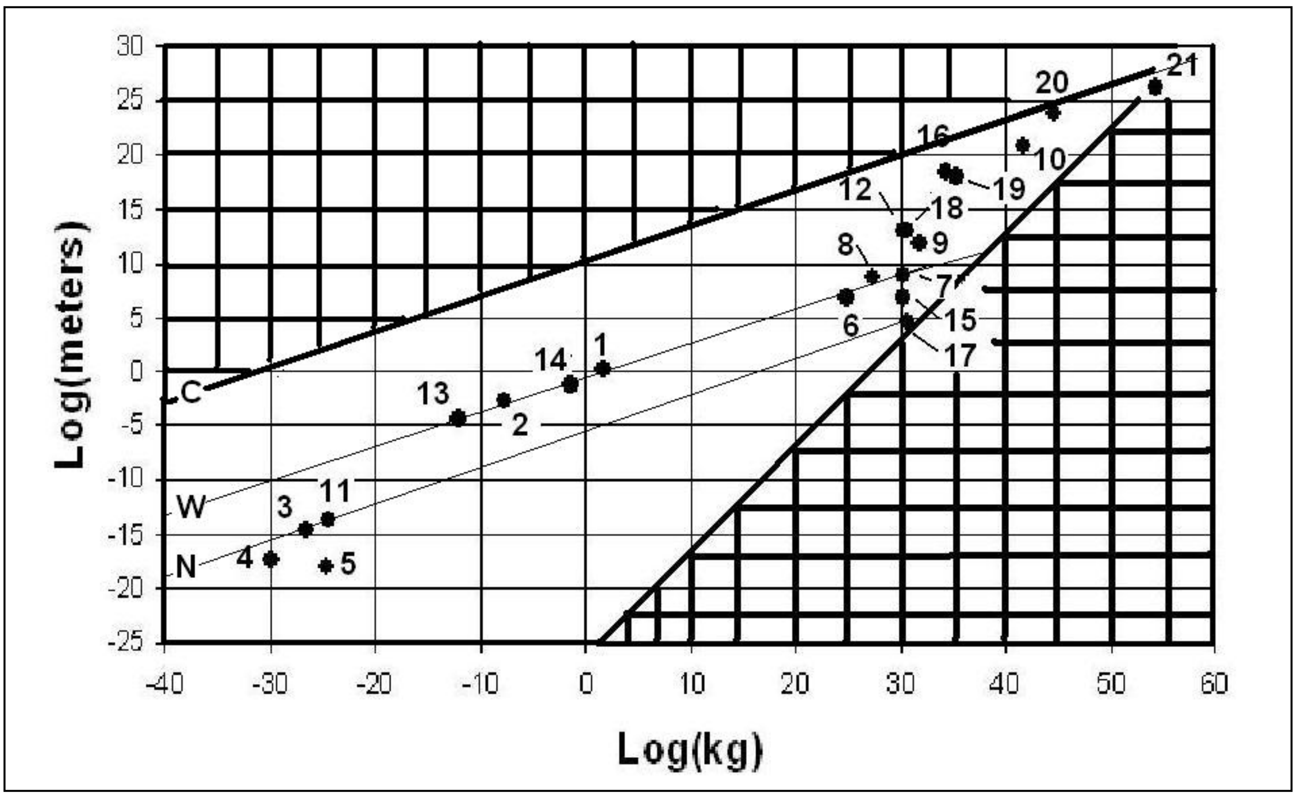
	Object	R (meters)	M (kg)
1	You	2.0	60
2	Mosquito	2×10^{-3}	2×10^{-6}
3	Proton	2×10^{-15}	2×10^{-27}
4	Electron	4×10^{-18}	1×10^{-30}
5	Z boson	1×10^{-18}	2×10^{-25}
6	Earth	6×10^{6}	6×10^{24}
7	Sun	1×10^{9}	2×10^{30}
8	Jupiter	4×10^{8}	2×10^{27}
9	Betelgeuse	8×10^{11}	6×10^{31}
10	Milky Way galaxy	1×10^{21}	5×10^{41}
11	Uranium atom	2×10^{-14}	4×10^{-25}
12	Solar system	1×10^{13}	2×10^{30}
13	Ameba	6×10^{-5}	1×10^{-12}
14	100-watt bulb	5×10^{-2}	5×10^{-2}
15	Sirius B white dwarf.	6×10^{6}	2×10^{30}
16	Orion nebula	3×10^{18}	2×10^{34}
17	Neutron star	4×10^{4}	4×10^{30}
18	Binary star system	1×10^{13}	4×10^{30}
19	Globular cluster M13	1×10^{18}	2×10^{35}
20	Cluster of galaxies	5×10^{23}	5×10^{44}
21	Entire visible universe	2×10^{26}	2×10^{54}

Answer Key

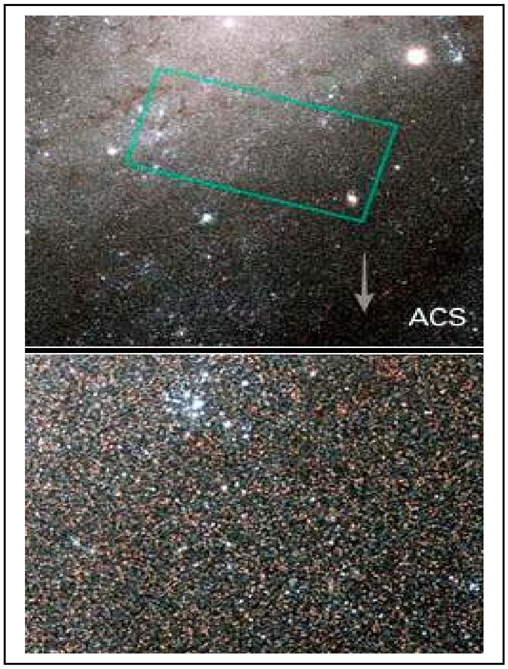
The figure below shows the various items plotted, and excluded regions cross-hatched. Students may color or shade-in the permitted region. This wedge represents all of the known objects and systems in our universe; a domain that spans a range of 85 orders of magnitude (10^{85}) in mass and 47 orders of magnitude (10^{47}) in size!

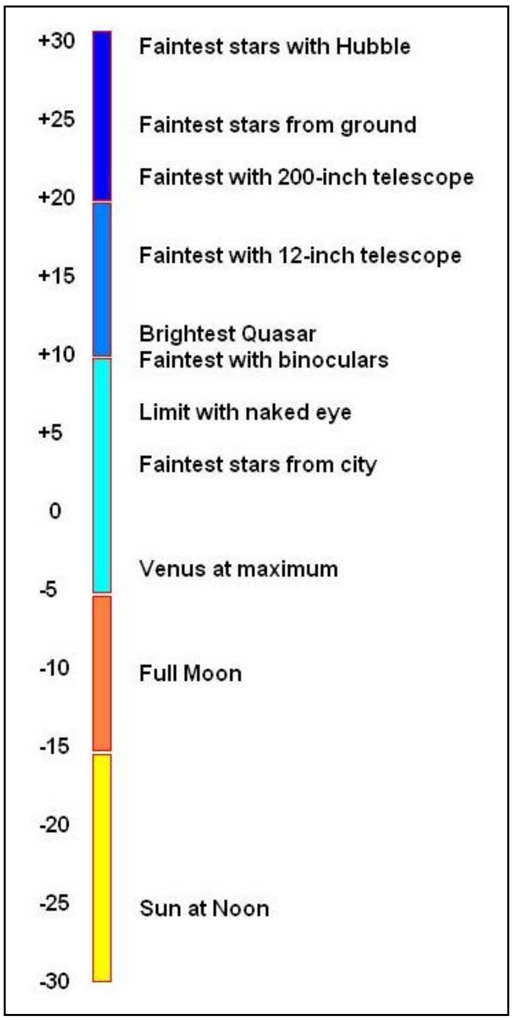
.

Inquiry: Can you or your students come up with more examples of objects or system that occupy some of the seemingly 'barren' regions of the permitted area?



Properties of Logarithms





Astronomers measure the brightness of a star in the sky using a magnitude scale. On this scale, the brightest objects have the SMALLEST number and the faintest objects have the LARGEST numbers. It's a 'backwards' scale that astronomers inherited from the ancient Greek astronomer Hipparchus.

The image to the left taken by the Hubble Space Telescope of individual stars in the galaxy NGC-300. The faintest stars are of magnitude +20.0.

1 – At its brightest, the planet Venus has a magnitude of -4.6. The faintest star you can see with your eye has a magnitude of +7.2. How much brighter is Venus than the faintest visible star?

2 – The full moon has a magnitude of -12.6 while the brightness of the sun is about -26.7. How many magnitudes fainter is the moon than the sun?

3 – The faintest stars seen by astronomers with the Hubble Space Telescope is +30.0. How much fainter are these stars than the sun?

4 - Jupiter has a magnitude of –2.7 while its satellite, Callisto, has a magnitude of +5.7. How much fainter is the Callisto than Jupiter?

5 – Each step by 1 unit in magnitude equals a brightness change of 2.5 times. A star with a magnitude of +5.0 is 2.5 times fainter than a star with a magnitude of +4.0. Two stars that differ by 5.0 magnitudes are 100-times different in brightness. If Venus was observed to have a magnitude of +3.0 and the full moon had a magnitude of -12.0, how much brighter was the moon than Venus?

Answer Key

1 – At its brightest, the planet Venus has a magnitude of -4.6. The faintest star you can see with your eye has a magnitude of +7.2. How much brighter is Venus than the faintest visible star?

Answer: +7.2 – (-4.6) = +7.2 + 4.6 = **+11.8 magnitudes**

2 – The full moon has a magnitude of -12.6 while the brightness of the sun is about -26.7. How many magnitudes fainter is the moon than the sun?

Answer: -12.6 – (-26.7) = -12.6 + 26.7 = **+14.1 magnitudes fainter.**

3 – The faintest stars seen by astronomers with the Hubble Space Telescope is +30.0. How much fainter are these stars than the sun?

Answer: +30.0 – (-26.7) = +30.0 + 26.7 = **+56.7 magnitudes fainter.**

4 - Jupiter has a magnitude of –2.7 while its satellite, Callisto, has a magnitude of +5.7. How much fainter is the Callisto than Jupiter?

Answer: +5.7 – (-2.7) = +5.7 + 2.7 = **+8.4 magnitudes fainter than Jupiter.**

5 – Each step by 1 unit in magnitude equals a brightness change of 2.5 times. A star with a magnitude of +5.0 is 2.5 times fainter than a star with a magnitude of +4.0. Two stars that differ by 5.0 magnitudes are 100-times different in brightness. If Venus was observed to have a magnitude of +3.0 and the full moon had a magnitude of -12.0, how much brighter was the moon than Venus?

Answer: The magnitude difference between them is +15.0, since every 5 magnitudes is a factor of 100 fainter, +15.0 is equivalent to 100x100x100 = 1 million times, so the moon is **1 million times brighter** than Venus.

Stars come in all different brightnesses and distances, which makes the sky very complicated in appearance.

Two quantities determine how bright a star will appear in the sky. The first is its distance, and the second is the brilliance or 'luminosity' of the star, measured in watts.

If you take a 100-watt bulb and place it 10 meters away from you, the amount of light you see will look the same as a 1-watt bulb only 1 meter away.

For stars, the apparent brightness or 'magnitude' of a star depends on its distance and its luminosity, also called its absolute magnitude. What you see in the sky is the apparent brightness of a star. The actual amount of light produced by the surface of the star is its absolute magnitude. A simple equation, basic to all astronomy, relates the star's distance in parsecs, D, apparent magnitude, m, and absolute magnitude, M as follows:

$$M = m + 5 - 5\log(D)$$

Problem 1 – The star Sirius has an apparent magnitude of m = -1.5, while Polaris has an apparent magnitude of m = +2.3, if the absolute magnitude of Sirius is M = +1.4 and Polaris is M = -4.6, what are the distances to these two stars?

Problem 2 – An astronomer determined the distance to the red supergiant Betelgeuse as 200 parsecs. If its apparent magnitude is m = +0.8, what is the absolute magnitude of this star?

Problem 3 – As seen in the sky, Regulus and Deneb have exactly the same apparent magnitudes of m = +1.3. If the distance to Deneb is 500 parcecs, and the absolute magnitude of Regulus is 1/9 that of Deneb, what is the distance to Regulus?

Problem 1 – The star Sirius has an apparent magnitude of m = -1.5, while Polaris has an apparent magnitude of m = +2.3, if the absolute magnitude of Sirius is M = +1.4 and Polaris is M = -4.6, what are the distances to these two stars?

Answer:

Sirius: +1.4 = -1.5 + 5 - 5logD
Log D = 2.9/5
LogD = 0.42 so the distance to Sirius is **D = 2.6 parsecs**

Polaris: -4.6 = +1.4 + 5 – 5LogD
LogD = +11.0/5
LogD = +2.2 so the distance to Polaris is **D = 158 parsecs**.

Problem 2 – An astronomer determined the distance to the red supergiant Betelgeuse as 200 parsecs. If its apparent magnitude is m = +0.8, what is the absolute magnitude of this star?

Answer: M = m + 5 - 5log(D)
= +0.8 + 5 – 5 log(200)
= +0.8 + 5 – 5(2.3) so for Betelgeuse **M = - 4.2**

Problem 3 – As seen in the sky, Regulus and Deneb have exactly the same apparent magnitudes of m = +1.3. If the distance to Deneb is 500 parcecs, and the absolute magnitude of Regulus is 1/9 that of Deneb, what is the distance to Regulus?

Answer: First find the absolute magnitude, M, for Deneb, then solve for D in the equation for Regulus:

Deneb: m = +1.3 and D = 500 parsecs then
M = +1.3 + 5 - 5log(500) so M = -7.2

Regulus: m = +1.3 and
M = 1/9 (-7.2) = -0.8 then
-0.8 = +1.3 + 5 – 5logD
LogD = +1.4 so for Regulus, **D = 25 parsecs**.

Solving exponential and logarithmic equations

Stars come in all different brightnesses and distances, which makes the sky very complicated in appearance.

Astronomers use a logarithmic scale to determine the brightness of stars as they appear in the sky. Called 'apparent magnitude' , this scale is a historical holdover from ancient star catalogs that ranked stars by their brightness. A First Ranked star with m = +1.0 is brighter than a Second Ranked star with m = +2.0 and so on.

The stellar magnitude scale, m, can be defined by a simple base-10 formula that defines the brightness of a star, B(m) as

$$B(m) = 10^{\mathbf{-0.4m}}$$

For example, Polaris the 'North Star' has an apparent magnitude of m = +2.3 and a brightness of B(+2.3) = 10-0.4(2.3) or B(2.3) = 0.12 on this scale. The star Sirius has an apparent magnitude of m = -1.4 and a brightness of B(-1.4) = 3.63.

Problem 1 – An astronomer measures the light from two identical stars in a binary system that are close together. If the brightness of the entire binary system is 0.008, what is the apparent magnitude of each of the individual stars?

Problem 2 - An astronomer measures the magnitudes of two stars and finds them to be exactly a factor of 100 in brightness. What is the apparent magnitude difference between these two stars?

Problem 3 - The Sun has an apparent magnitude of – 26.5 while the faintest star that can be seen by the Hubble Space Telescope has an apparent magnitude of +31. By what factor is the Sun brighter than the faintest star seen by Hubble?

Problem 1 – An astronomer measures the light from two identical stars in a binary system that are close together. If the brightness of the entire binary system is 0.008, what is the apparent magnitude of each of the individual stars?

Answer: Because the stars are identical, the brightness of each star is just B = 0.008/2 = 0.004. Then solve

B(m) = 0.004 to determine m for each of the stars individually.
$0.004 = 10^{\mathbf{-0.4m}}$
Log(0.004) = -0.4m
-2.4 = -0.4m

So the apparent magnitude of each star, individually, is **m = +6.0**

Problem 2 - An astronomer measures the magnitudes of two stars and finds them to be exactly a factor of 1/100 in brightness. What is the apparent magnitude difference between these two stars?

Answer: B(m) = 1/100
so $1/100 = 10^{\mathbf{-0.4m}}$
Log(1/100) = -0.4m
-2 = -0.4m
m = 5
The stars differ in apparent magnitude by exactly **5.0 magnitudes**.

Problem 3 - The Sun has an apparent magnitude of – 26.5 while the faintest star that can be seen by the Hubble Space Telescope has an apparent magnitude of +31. By what factor is the Sun brighter than the faintest star seen by Hubble?

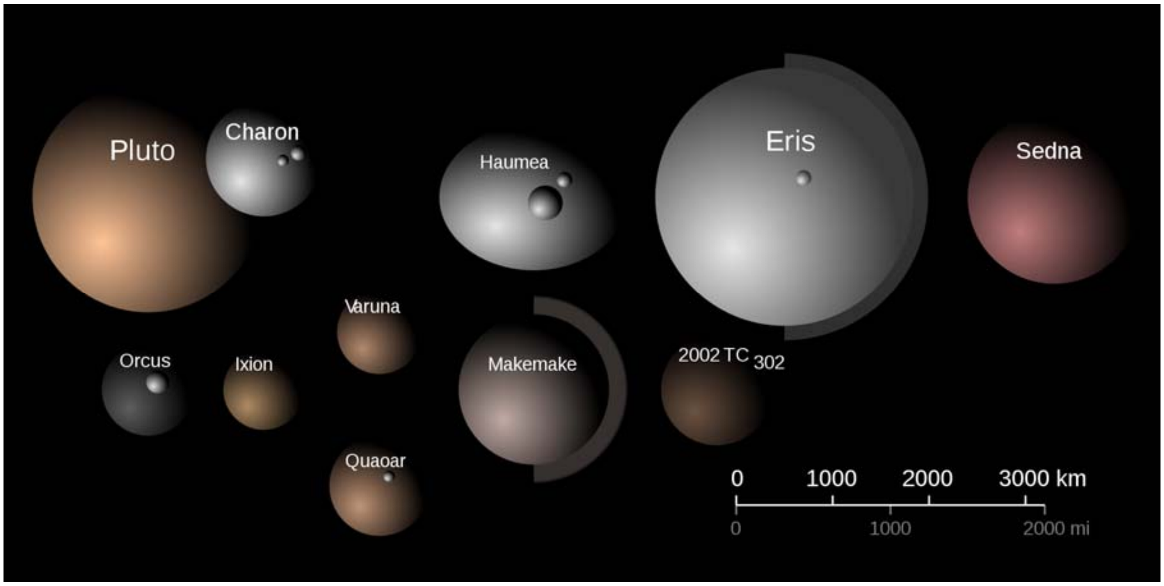
Answer:

$B(-26.5) = 10^{\mathbf{-0.4(-26.5)}}$ so $B(-26.5) = 4.0 \times 10^{10}$

$B(+31) = 10^{\mathbf{-0.4(+31)}}$ so $B(+31) = 4.0 \times 10^{-13}$

So B(sun)/B(star) = $4.0 \times 10^{10} / 4.0 \times 10^{-13}$ or a factor of 10^{23} times.

In other words, the Sun is 100,000,000,000,000,000,000,000 times brighter than the faintest stars seen by the Hubble!



Object	Distance (AU)	Period (years)
Mercury	0.4	0.24
Venus	0.7	0.61
Earth	1.0	1.0
Mars	1.5	1.88
Ceres	2.8	4.6
Jupiter	5.2	11.9
Saturn	9.5	29.5
Uranus	19.2	84.0
Neptune	30.1	164.8
Pluto	39.4	247.7
Ixion	39.7	
Huya	39.8	
Varuna	42.9	
Haumea	43.3	285
Quaoar	43.6	
Makemake	45.8	310
Eris	67.7	557
1996-TL66	82.9	
Sedna	486.0	

Astronomers have detected over 500 bodies orbiting the sun well beyond the orbit of Neptune. Among these 'Trans-Neptunian Objects (TNOs) are a growing number that rival Pluto in size. This caused astronomers to rethink how they should define the term 'planet'.

In 2006 Pluto was demoted from a planet to a dwarf planet, joining the large asteroid Ceres in that new group. Several other TNOs also joined that group, which now includes five bodies shown highlighted in the table. A number of other large objects, called Plutoids, are also listed.

Problem 1 - From the tabulated data, graph the distance as a function of period on a calculator or Excel spreadsheet. What is the best-fit: A) Polynomial function? B) Power-law function?

Problem 2 - Which of the two possibilities can be eliminated because it gives unphysical answers?

Problem 3 - Using your best-fit model, what would you predict for the periods of the TNOs in the table?

Problem 1 - From the tabulated data, graph the distance as a function of period on a calculator or Excel spreadsheet. What is the best-fit:

A) Polynomial function? **The N=3 polynomial $D(x) = -0.0005x^3 + 0.1239x^2 + 2.24x - 1.7$**

B) Power-law function? **The N=1.5 powerlaw: $D(x) = 1.0x^{1.5}$**

Problem 2 - Which of the two possibilities can be eliminated because it gives unphysical answers? The two predictions are shown in the table:

Object	Distance	Period	N=3	N=1.5
Mercury	0.4	0.24	-0.79	0.25
Venus	0.7	0.61	-0.08	0.59
Earth	1	1	0.66	1.00
Mars	1.5	1.88	1.93	1.84
Ceres	2.8	4.6	5.53	4.69
Jupiter	5.2	11.9	13.22	11.86
Saturn	9.5	29.5	30.33	29.28
Uranus	19.2	84	83.44	84.13
Neptune	30.1	164.8	164.34	165.14
Pluto	39.4	247.7	248.31	247.31
Ixion	39.7		251.21	250.14
Huya	39.8		252.19	251.09
Varuna	42.9		282.94	280.99
Haumea	43.3	285	286.99	284.93
Quaoar	43.6		290.05	287.89
Makemake	45.8	310	312.75	309.95
Eris	67.7	557	562.67	557.04
1996-TL66	82.9		750.62	754.80
Sedna	486		-27044.01	10714.07

Answer: The N=3 polynomial gives negative periods for Mercury, Venus and Sedna, and poor answers for Earth, Mars, Ceres and Jupiter compared to the N=3/2 power-law fit. The N=3/2 power-law fit is the result of Kepler's Third Law for planetary motion which states that the cube of the distance is proportional to the square of the period so that when all periods and distances are scaled to Earth's orbit, Period = $\text{Distance}^{3/2}$

Problem 3 - See the table above for shaded entries

Because of friction with Earth's atmosphere, satellites in Low Earth Orbit below 600 kilometers, experience a gradual loss of orbit altitude over time. The lower the orbit, the higher is the rate of altitude loss, and it can be approximated by the formula:

$$T(h) = 0.012De^{0.025(h-150)}$$

where h is the altitude of the orbit in kilometers above Earth's surface, and T(h) is in days until re-entry.

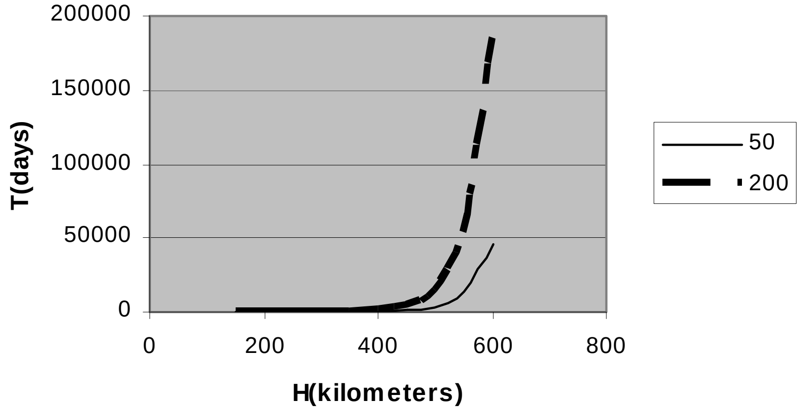
The variable, D, is called the ballistic coefficient and is a measure of how massive the satellite is compared to the surface area facing its direction of motion (in kilograms/meter^2).

Problem 1 - Graph this exponential function for a domain of satellite orbits given by 150 kilometers $<$ h $<$ 600 kilometers for D = 50 kg/m^2 and D = 200 kg/m^2.

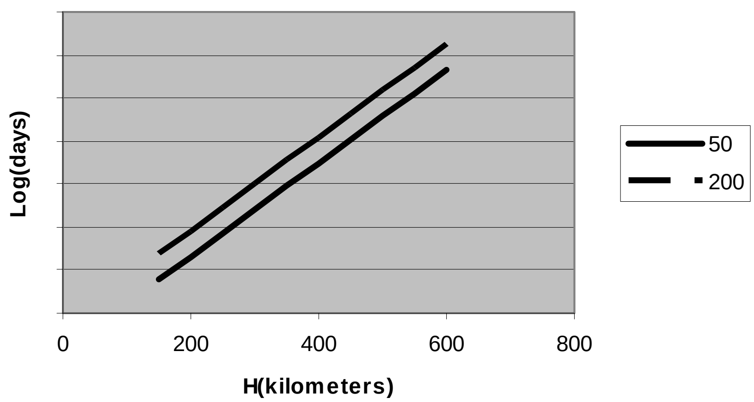
Problem 2 - Graph this function for the same domain and values for D as a log-linear plot: h vs log(T).

Problem 3 - Suppose that the Hubble Space Telescope, D=11 kg/m^2, is located in an orbit with an altitude of 575 kilometers following the 're-boost' provided by the Space Shuttle crew during the last Servicing Mission in 2009. The Space Shuttle raised the orbit of HST by 10 kilometers. A) By what year would HST have re-entered had this re-boost not occurred? B) About when will the HST re-enter the atmosphere following this Servicing Mission?

Problem 1 - Graph this exponential function for a domain of satellite orbits given by 150 kilometers < h < 600 kilometers for D = 50 kg/m^2 and D = 200 kg/m^2 .



Problem 2 - Graph this function for the same domain and values for D as a log-linear plot: h vs log(T).



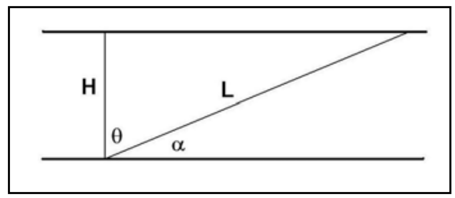
Problem 3 - Answer:

A) Before the re-boost, the altitude was 575 - 10 km = 565 km, in 2009, so T = 0.012(11)e(0.025 (565-150)) = 4,230 days or 11 years from 2009 so the reentry would have occurred **around 2020**.

B) After the re-boost, the altitude was 575 km in 2009, so T = 0.012(11)e(0.025 (575-150)) = 5,432 days or 15 years from 2009 so the reentry occurs around **2024**.

Imagine that the atmosphere was a thick blanket of gas. As you look straight up, you can see the stars, but as you look towards the horizon, the starts fade away completely.

A very simple geometric below, shows just how this happens. The parallel lines represent the top and bottom of the atmosphere. H is the thickness of the atmosphere 'straight up' towards the Zenith, and L is the length of a sight line through the atmosphere tilted at an angle, theta, from the Zenith direction.



Problem 1 - What is the relationship between the zenith angle, θ, and the elevation angle, α, where H is perpendicular to the parallel lines?

Problem 2 - The thickness of the atmosphere is assumed to be fixed. What is the length, L, in terms of H and θ?

Problem 3 - What is the length, L, in terms of H and the elevation angle α?

Problem 4 - For what angles, θ and α, will the path through the atmosphere equal 2H?

Problem 5 - The brightness of a star is given by $I(L) = I_0 e^{-\frac{L}{b}}$ where I_0 is the brightness of the star in the zenith direction, and b is the path length through the atmosphere for which the brightness of the star will dim by a factor of exactly e^{-1} = 0.37. If b = H, and I_0 = 2, graph the function I(L) for the domain L:[H, 3H] and state its range.

Problem 1 - Answer: θ **= 90 -** α**.**

Problem 2 - Answer: Since L and H are the sides of a right triangle, H = Lcos(θ) so

$$L = \frac{H}{\cos(\theta)} = H \text{ Secant}(\theta)$$

Problem 3 - Answer: H = L sin(α) so

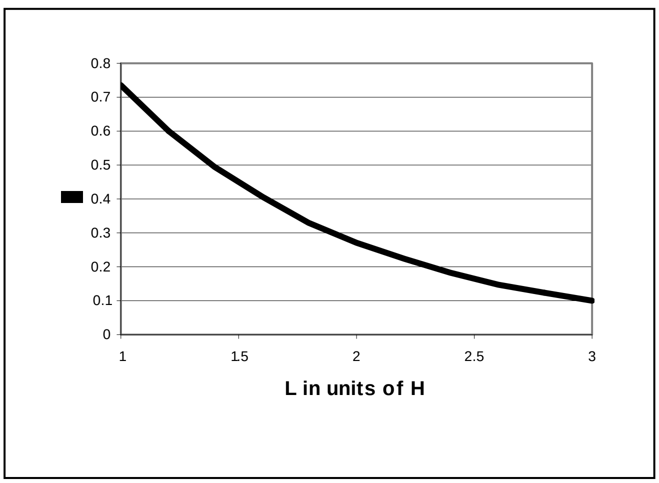
$$L = \frac{H}{\sin(\alpha)} = H \text{ Cosecant}(\alpha)$$

Problem 4 - Answer: 2H = H Secant(θ) so
Secant(θ) = 2 so
Cos(θ) = 0.5 and θ **= 60 degrees**.

Since θ = 90 - α,
α = 90 - 60,
α**=30 degrees**.

So, for an elevation angle of α=30 degrees above the horizon, the path through the atmosphere is twice the zenith distance, H.

Problem 5 - Answer: For values in this domain, the exponential term, -L/b will be from -1 to -3 so the range of F(L) will be from 0.37(2)= 0.74 to 0.05(2)= 0.10 so **I: [0.10, 0.74]** and the **graph is shown below**.



Modeling with Power Functions

Time (sec)	Log(Brightness) ($erg/sec/cm^2$)
200	-10.3
500	-10.7
1,000	-11.0
6,000	-11.7
10,000	-12.0
25,000	-12.3
100,000	-13.0
500,000	-13.8

Gamma-ray bursts, first spotted in the 1960's, occur about once every day, and are believed to be the dying explosions from massive stars being swallowed whole by black holes that form in their cores, hours before the explosion. The amount of energy released is greater than entire galaxies of starlight.

This burst began January 16, 2005, and lasted 529,000 seconds as seen by the Swift satellite's X-ray telescope. The data for GRB 060116 is given in the table to the left. This source, located in the constellation Orion, but is over 10 billion light years behind the Orion Nebula!

Problem 1 - Plot the tabulated data on a graph with x = Log(seconds) and y = Log(Brightness).

Problem 2 - What is the best-fit linear equation that characterizes the data over the domain x: [2.0, 5.0]?

Problem 3 - What is the equivalent power-law function that represents the linear fit to the data?

Problem 4 - If the Gamma-ray Burst continues to decline at this rate, what will be the brightness of the source by A) February 16, 2005? B) January 16, 2006?

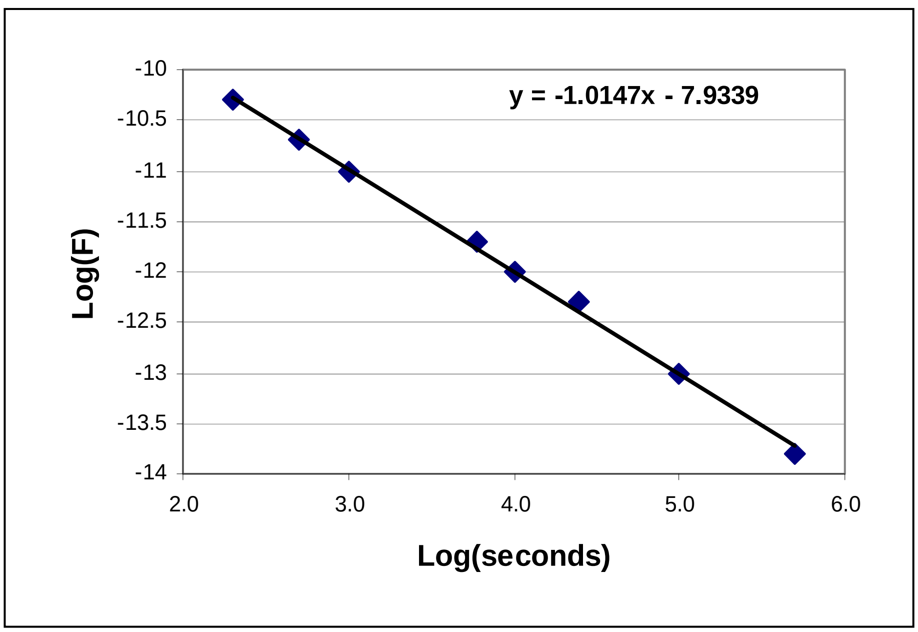
Problem 1 - Answer: See figure below.

Problem 2 - Answer: See figure below with **y = -1.0x - 7.93**

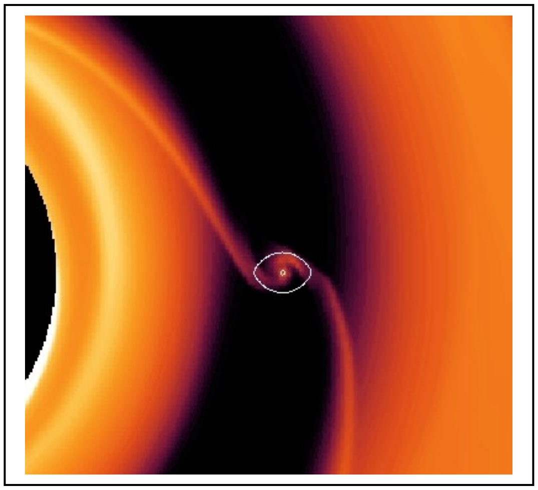
Problem 3 - Answer: LogB = -1.0Logt - 7.93 so 10LogB = 10(-1.0Logt - 7.93) or **$B(t) = 1.17x10^{-8}\ t^{-1.0}$**

Problem 4 - Answer: A) First calculate the number of seconds elapsed between January 16 and February 16 which equals 31 days or 31 x (24 hrs) x (3600 sec/hr) = 2,678,400. Then $B(t) = 1.17x10^{-8}\ (2678400)^{-1.0}$ and so **$B(t) = 4.37 \times 10^{-15}$ ergs/sec/cm^2**. B) The elapsed time is 365 days or 3.1×10^7 seconds so $B(t) = 1.17x10^{-8}\ (3.1x10^7)^{-1.0}$ and so **$B(t) = 3.77 \times 10^{-16}$ ergs/sec/cm^2**.

Note: Research reported by En-Wei Liang in October 24, 2009 article 'A comprehensive analysis of Swift/XRT data' (Astro-ph.HE: arXive:0902.3504v2). The study of over 400 GRBs found 19 that had power-law light curves out to 100,000 seconds and more.



Logistics Growth Functions



The growth of planets from the limited materials in the orbiting disk of dust and gas can be approximated by a logistics function.

Because of the way in which orbiting material moves, material outside the orbit of the planet travels more slowly than material inside the planet's orbit. Eventually, the forming planet consumes the material along its orbit and forms an ever-expanding gap. Eventually the process stops when no more gas exists to be captured.

The growth of Earth can be approximated by the 'accretion' function

$$M(t) = \frac{6000}{1 + 400e^{-\frac{t}{5}}}$$

where its mass is in units of 10^{21} kilograms and the elapsed time, t, is in millions of years.

Problem 1 – Graph the accretion function over the domain t:[0,70]

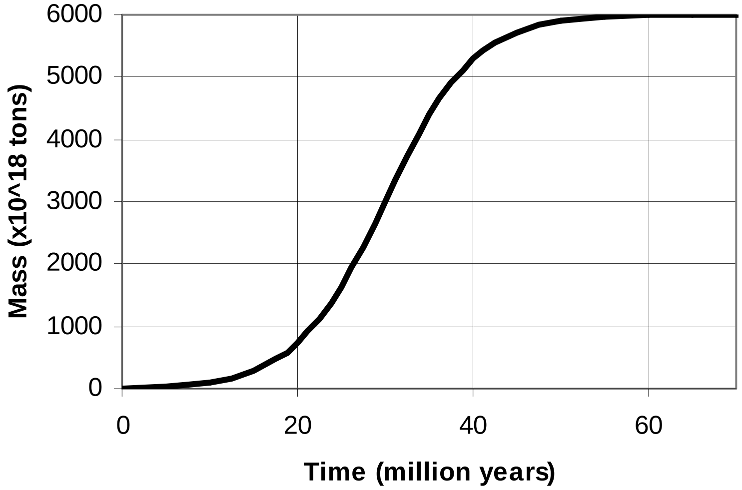
Problem 2 – What is the asymptotic (limiting) mass of Earth in kilograms?

Problem 3 - At what mass, in kilograms, does Earth begin the accretion process as a planetoid?

Problem 4 - How long does the function predict that it took Earth to reach 95% of its final mass?

Problem 5 - At what rate is the mass increasing near a time of 30 million years in units of 10^{18} tons per million years?

Problem 1 – Answer:



Problem 2 – Answer: 6000 x 10^{18} tons or **6.0 x 10^{24} kg**.

Problem 3 - Answer: At t=0 M(0) = 6000 / (1 + 400) so **M = 1.49 x 10^{22} kilograms**

Problem 4 - Answer: M(t) = 0.95x6000 = 5700 so

5700 = 6000 / (1 + 400 $e^{(-t/5)}$) solve for t.
0.053 = 400 $e^{(-t/5)}$
ln(1.31 x 10^{-4}) = -t/5 so **t = 45 million years**.

Problem 5 - At what rate is the mass increasing near a time of 30 million years in units of 10^{18} tons per million years?

Answer: Evaluate M(t) for two times near 30 million years:
M(25) = 1624 x 10^{18} tons
M(35) = 4396 x 10^{18} tons.

The slope = rate of change
= (M(35)-M(25)/10 million years
= **2.78 x 10^{20} tons/million years**