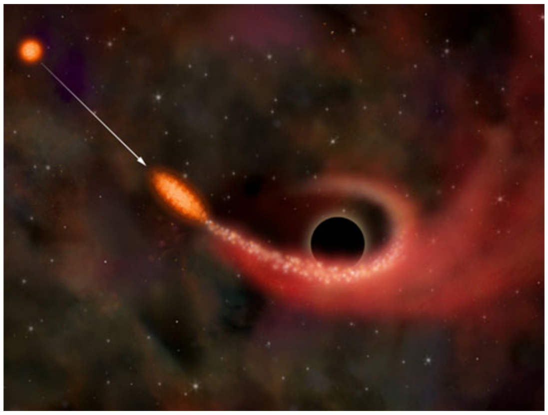
N-th Roots and Rational Exponents



A star passing too close to a black hole will be torn to shreds by the black holes intense gravity field, as shown in this artistic painting. (Courtesy M.Weiss NASA/Chandra)

One of the most peculiar things about black holes is that, when you are close to one, time and space are badly distorted.

Imagine two astronauts, Stan and Sharon, each with a synchronized clock. Stan remains at a great distance from the black hole, but Sharon takes a trip close to the black hole. Although the passage of time measured by Stan's clock will seem normal, he will watch as the reading on Sharon's clock slows down as she gets closer to the black hole!

Problem 1 – The formula that relates the elapsed minutes on Sharon's clock, x, to the time that Stan sees passing on her clock, y, when Sharon is at a distance of r kilometers from the center of the black hole is given by:

$$y = \frac{x}{\sqrt{1 - \frac{2.8}{r}}}$$

If Sharon is in obit around the black hole at a distance of r = 2800 kilometers; A) How many hours will elapse on Stan's clock for every hour, x, that passes on Sharon's clock? B) How many seconds is the time difference between the two cocks?

Problem 2 – Sharon travels to within 4 kilometers of the black hole, without being torn to shreds by its enormous gravity. A) How many hours will elapse on Stan's clock for every hour, x, that passes on Sharon's clock? B) How many seconds is the time difference between the two cocks?

Problem 3 – To five significant figures, how close does Sharon have to be to the black hole before one week elapses on Stan's clock for every hour that passes on Sharon's clock?

Problem 1 – The formula that relates the elapsed minutes on Sharon's clock, x, to the time that Stan sees passing on her clock, y, is given by:

$$y = \frac{x}{\sqrt{1 - \frac{2.8}{r}}}$$

If Stan and Sharon are together in obit around the black hole at a distance of r = 2800 kilometers, A) how many hours will elapse on Stan's clock for every hour, x, that passes on Sharon's clock? B) how many seconds is the time difference between the two cocks?

Answer: A) r = 2800 so y = $x/(1-0.001)^{1/2}$ = 1.0005 x so for x = 1 hour on Sharon's clock, Stan will see y=**1.0005 hours** pass.

B) This equals a time difference between them of y-x = (1.0005-1.0)*3600 = **1.8 seconds**.

Problem 2 – Sharon travels to within 4 kilometers of the black hole, without being torn to shreds by its enormous gravity. Recalculate your answers to Problem 1 at this new distance.

Answer: A) r = 4 so y = $x/(1-0.7)^{1/2}$ = 1.83 x so for x = 1 hour on Sharon's clock, Stan will see **1.83 hours** pass.

B) This equals an additional y-x = (1.83 -1.0)*3600 = **2,988 seconds**.

Problem 3 – To five significant figures, how close does Sharon have to be to the black hole before one week elapses on Stan's clock for every hour that passes on Sharon's clock?

Answer: One week = 24 hours/day x 7 days/week = 168 hours, so we want to find a value for r such that y = 168 for x = 1.

$168 = 1/(1-2.8/r)^{1/2}$ so solving for r we get

$$r = \frac{2.8}{1 - \left(\frac{1}{168}\right)^2}$$

so r = **2.8001 kilometers.**

Note: For this problem, the black hole's radius is exactly 2.8 kilometers, so Sharon is within 0.0001 kilometers or 100 centimeters of its surface!

The lovely nebulae that astronomers photograph in all of their vivid colors are created by the ultraviolet light from very hot stars. The intensity of this light causes hydrogen gas to become ionized within a spherical zone defined by the equation;

$$R = 0.3L^{\frac{1}{2}}N^{-\frac{2}{3}}$$

where N is the density of the gas in atoms/cm^3 and L is the luminosity of the stars in multiples of the sun's power. And R is the radius of the nebula in light years.

The image above was taken of the famous Great Nebula in Orion (Messier-42) by the Hubble Space Telescope. Notice its semi-circular appearance.

Problem 1 – Solve the equation for the luminosity of the stars, L, given the gas density and nebula radius.

Problem 2 - The Orion Nebula has a radius of R=2.5 light years, and an average density of about N=60 atoms/cm^3. To two significant figures, what is the total luminosity, L, of the stars providing the energy to keep the nebula 'turned on'?

Problem 3 – Solve the equation for the gas density, given the luminosity of the stars and the radius of the nebula.

Problem 4 – The Cocoon Nebula has a radius of R=3 light years and is produced by a star with a luminosity of L = 1000 times the sun. To two significant figures, what is the approximate gas density, N, in the nebula?

Problem 1 – Solve the equation for the luminosity of the stars, L, given the gas density and nebula radius.

Answer: $\mathbf{L = 900\ r^2\ N^{4/3}}$

Problem 2 - The Orion Nebula has a radius of R=2.5 light years, and an average density of about N=60 atoms/cm^3. To two significant figures, what is the total luminosity, L, of the stars providing the energy to keep the nebula 'turned on'?

Answer: $L = 900\ (2.5)^2\ (60)^{4/3}$
L = 1,300,000 times the sun.

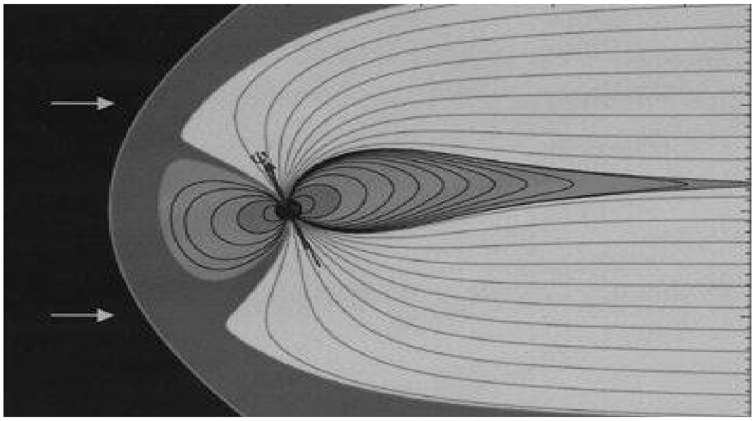
Problem 3 – Solve the equation for the gas density, given the luminosity of the stars and the radius of the nebula.

Answer: $\mathbf{N = 0.16\ L^{3/4}\ R^{-3/2}}$

Problem 4 – The Cocoon Nebula has a radius of R=3 light years and is produced by a star with a luminosity of L = 1000 times the sun. To two significant figures, what is the approximate gas density in the nebula?

Answer: $N = 0.16\ (1000)^{3/4}\ (3)^{-3/2}$

= **5.5 atoms/cm^3**



When the solar wind flows past Earth, it pushes on Earth's magnetic field and compresses it. The distance from Earth's center, R, where the pressure from Earth's magnetic field balances the pressure of the solar wind is given by the equation:

$$R^6 = \frac{0.72}{8\pi D V^2}$$

In this equation, D is the density in grams per cubic centimeter (cc) of the gas (solar wind, etc) that collides with Earth's magnetic field, and V is the speed of this gas in centimeters per second. The quantity, R, is the distance from the center of Earth to the point where Earth's magnetic field balances the pressure of the solar wind in the direction of the sun.

Problem 1 - The table below gives information for five different solar storms. Complete the entries to the table below, rounding the answers to three significant figures:

Problem 2 - The fastest speed for a solar storm 'cloud' is 1500 km/s. What must the density be in order that the magnetopause is pushed into the orbits of the geosynchronous communication satellites at 6.6 Re?

Storm	Date	Day Of Year	Density (particle/cc)	Speed (km/s)	R (km)
1	11/20/2003	324	49.1	630	
2	10/29/2003	302	10.6	2125	
3	11/06/2001	310	15.5	670	
4	3/31/2001	90	70.6	783	
5	7/15/2000	197	4.5	958	

The information about these storms and other events can be obtained from the NASA ACE satellite by selecting data for H* density and V_x (GSE)

http://www.srl.caltech.edu/ACE/ASC/level2/lvl2DATA_MAG-SWEPAM.html

Storm	Date	Day Of Year	Density (particle/cc)	Speed (km/s)	R (km)
1	11/20/2003	324	49.1	630	**42,700**
2	10/29/2003	302	10.6	2125	**37,000**
3	11/06/2001	310	15.5	670	**51,000**
4	3/31/2001	90	70.6	783	**37,600**
5	7/15/2000	197	4.5	958	**54,800**

Problem 2 - The fastest speed for a solar storm 'cloud' is 3000 km/s. What must the density be in order that the magnetopause is pushed into the orbits of the geosynchronous communication satellites at 6.6 Re (42,000 km)?

Answer: Solve the equation for D to get:

$$D = \frac{0.72}{8\pi R^6 V^2}$$

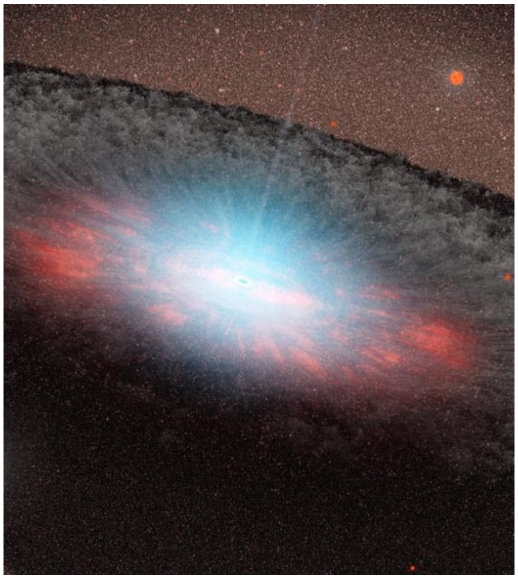
For 1500 km/s V = 1.5 x 10^8 cm/s, and for R = 6.6, we have

$$D = 0.72/(8 \times 3.14 \times 6.6^6 \times (1.5 \times 10^8)^2) = 1.52 \times 10^{-23} \text{ gm/cc}$$

Since a proton has a mass of 1.6 x 10^{-24} grams, this value for the density, D, is equal to (1.52 x 10^{-23}/ 1.6 x 10^{-24}) = 9.5 protons/cc.

For Extra Credit, have students compute the density if the solar storm pushed the magnetopause to the orbit of the Space Station (about R = 1.01 RE). **Answer:** D = 3 x 10^{-19} gm/cc or 187,000 protons/cc. A storm with this density has never been detected, and would be catastrophic!

Properties of Rational Exponents



Artistic rendering of an accretion disk showing gas heating up as it flows in to the black hole. (Courtesy NASA/JPL – Caltech)

Matter that flows into a black hole usually takes up residence in an orbiting disk of gas called an accretion disk. Friction causes this gas to heat up, and the temperature of the gas is given by the formula:

$$T = 3.7x10^6 \frac{(MC)^{\frac{1}{4}}}{R^{\frac{3}{4}}}$$

where M is the mass of the black hole, C is the rate at which the gas enters the disk and R is the distance of the gas from the black hole.

To make calculations easier when using large astronomical numbers, astronomers specify M in multiples of the mass of our sun, R in multiples of the Earth-Sun distance, and C in terms of the number of solar masses consumed each year. So, for a 100 solar-mass black hole accreting matter at a rate of 0.0001 solar masses each year, the temperature at a distance of 10 Astronomical Units will be found by substituting R=10 ,M=100 and C=0.0001 into the equation.

Problem 1 – What does the formula look like for the case of T evaluated at C=0.001 solar masses per year, and R = 2 times the Earth-Sun distance?

Problem 2 – For a black hole with a mass of M=1000 times the sun, and consuming gas at a rate of C=0.00001 solar masses each year, how far from the black hole, in kilometers, will the gas be at 'room temperature' of T = 290 K? (The Earth-Sun distance equals R =1 AU = 150 million kilometers)

Problem 3 – Consider two black holes with masses of 1.0 times the sun, and 100.0 times the Sun, consuming gas at the same rate. An astronomer makes a temperature measurement at a distance of R=x from the small black hole, and a distance of y from the large black hole. A) What is the formula that gives the ratio of the temperatures that he measures in terms of x and y? B) What is the temperature ratio if the astronomer measures the gas at the same distance? C) For which black hole is the temperature of the accreting gas highest at each distance?

Problem 1 – What does the formula look like for the case of T evaluated at C=0.001 solar masses per year, and R = 2 times the Earth-Sun distance?

$T = 3.7x10^6\ M^{1/4}\ (0.001)^{1/4} / 2^{3/4}$ = **390,000 M   K**

Problem 2 – For a black hole with a mass of M=1000 times the sun, and consuming gas at a rate of C=0.00001 solar masses each year, how far from the black hole in kilometers will the gas be at 'room temperature' of T = 290 K? (The Earth-Sun distance equals 150 million kilometers)

$290 = 3.7x106\ (.00001)^{1/4}\ (1000)^{1/4} / R^{3/4}$

$R^{3/4} = 4035$

R = **64,060** times the Earth-Sun distance.

or   R = 64,060 x 150 million km = $9.6x10^{12}$ kilometers.

Note: 1 light year = 9.2×10^{12} km so you would have to be just over 1 light year from the black hole

Problem 3 – Consider two black holes with masses of 1.0 times the sun, and 100.0 times the sun, consuming gas at the same rate. An astronomer makes a temperature measurement at a distance of R=x from the small black hole, and a distance of y from the large black hole. A) What is the formula that gives the ratio of the temperatures that he measures in terms of x and y? B) What is the temperature ratio if the astronomer measures the gas at the same distance? C) For which black hole is the temperature highest at each distance?

A)

$T(x) = 3.7x10^6\ C^{1/4}\ (1.0)^{1/4} / x^{3/4}$

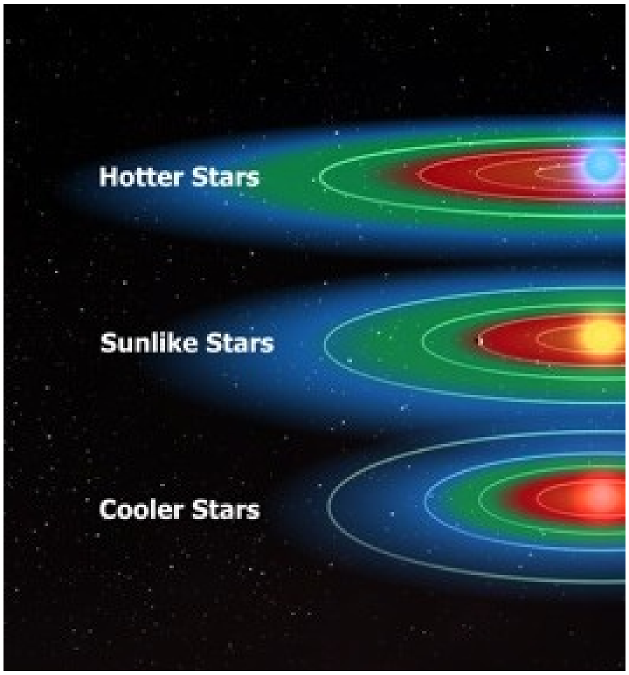
$T(y) = 3.7x10^6\ C^{1/4}\ (100)^{1/4} / y^{3/4}$

So $Tx/Ty = 3.7x10^6\ y^{3/4} / 1.2x10^7\ x^{3/4}$ **$Tx/Ty = 0.31\ y^{3/4} / x^{3/4}$**

B) **Tx/Ty = 0.31**.

C) Because T(x)/T(y) = 0.31, the more massive black hole, T(y), is 1/0.31 = **3.2 times** hotter! This is true at every distance because for x = y, the ratio of the temperatures is independent of x and y.

Properties of Rational Exponents



As more planets are being discovered beyond our solar system, astronomers are searching for planets on which liquid water can exist. This means that the planet has to be close enough for water to turn from solid ice to a liquid (T = 273 Kelvin) but not so hot that the liquid water turns to steam (T = 373 Kelvin).

Astronomers call this range the Habitable Zone around the star.

Sketch of Habitable Zones around stars of different temperatures and sizes. (Courtesy NASA/Kepler)

A formula relates the temperature of an Earth-like planet to its distance from its star, d, the radius of the star, R, and the temperature of its star, T*:

$$T = 0.6T^{*}\left(\frac{R}{d}\right)^{\frac{1}{2}}$$

where R and d are in kilometers, and T is the temperature in Kelvins.

Problem 1 – For a star identical to our sun, T* = 5770 K and R = 700,000 km. At what distance from such a star will a planet be warm enough for water to be in liquid form?

Problem 2 – The star Polaris has a temperature of 7,200 K and a radius 30 times larger than our sun.

A) Over what distance range will water remain in liquid form? (Note Astronomers call this the Habitable Zone of a star).

B) Compared to the Earth-Sun distance of 150 million km, called an Astronomical Unit, what is this orbit range in Astronomical Units?

Problem 1 – For a star identical to our sun, T* = 5770 K and R = 700,000 km. At what distance from such a star will a planet be warm enough for water to be in liquid form?

Answer: $273 = 0.6 (5770) (700{,}000/D)^{1/2}$

$D^{1/2} = 0.6 (5770)(700{,}000)^{1/2}/273$

$D^{1/2} = 10{,}601$

D = 112 million km.

Problem 2 – The star Polaris has a temperature of 7,200 K and a radius 30 times larger than our sun. A) Over what distance range will water remain in liquid form? (Note Astronomers call this the Habitable Zone of a star). B) Compared to the Earth-Sun distance of 150 million km, called an Astronomical Unit, what is this orbit range in Astronomical Units?

Answer:

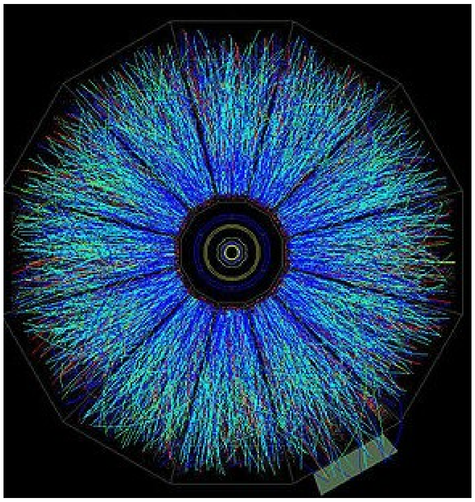
A)

$T = 1.9\times10^{7} / D^{1/2}$

$D^{1/2} = 1.9\times10^{7}/273$ squaring both sides, **$D = 4.8\times10^{9}$ km**

$D^{1/2} = 1.9\times10^{7}/373$ squaring both sides, **$D = 2.6\times10^{9}$ km**

B) **32 AU to 17 AU**. This is about the distance between the orbits of Neptune and Pluto in our solar system.



Seen here is the decay of a quark-gluon plasma shown by numerous particles streaming out from the site of the plasma. These conditions were created at the Brookhaven Heavy Ion Collider.

During the Big Bang, the universe was much hotter and denser than it is today. As it has continued to expand in time, the temperature continues to decrease in time.

The temperature can be predicted from a mathematical model of the expansion of the universe, and the properties of matter at various times in the universe's history.

A formula that relates the temperature, T, in degrees Kelvin to the elapsed time in seconds since the Big Bang, t, is:

$$t = \sqrt{\frac{c^2}{48\pi G a T^4}}$$

Problem 1 – The variables c, G and a are actually physical constants whose values are measured under laboratory conditions. The speed of light, c, has a value of 3×10^{10} centimeters/sec; the constant of gravity, G, has a value of 6.67×10^{-8} $cm^3/gm^2/sec^2$; and the radiation constant, a, has a value of 7.6×10^{-15} gm sec^2/cm^5 K^4. Based on these values, and using $\pi = 3.14$, re-write the formula so that the time, t, is expressed in seconds when T is expressed in degrees K.

Problem 2 – Derive a formula that gives the temperature, T, in terms of the time since the Big Bang, t, in seconds.

Problem 3 – A very important moment in the history of the universe occurred when the temperature of matter became so low that the electrons and protons in the expanding cosmic plasma cooled enough that stable hydrogen atoms could form. This happened at a temperature of about 4,000 K. How many years after the Big Bang did this occur?

Problem 1 – The variables c, G and a are actually physical constants whose values are measured under laboratory conditions. The speed of light, c, has a value of $3x10^{10}$ centimeters/sec; the constant of gravity, G, has a value of $6.67x10^{-8}$ centimeters^2/gram^2/sec^2; and the radiation constant, a, has a value of 7.6×10^{-15} grams sec^2/cm^5 K^4. Based on these values, and using π = 3.14, re-write the formula so that the time, t, is expressed in seconds when T is expressed in degrees K.

Answer:

$$t = \frac{1.08x10^{20}}{T^2} \text{ seconds}$$

Problem 2 – Derive a formula that gives the temperature, T, in terms of the time since the Big Bang, t, in seconds.

Answer:

$$T = 1.04x10^{10}\ t^{1/2} \text{ K}$$

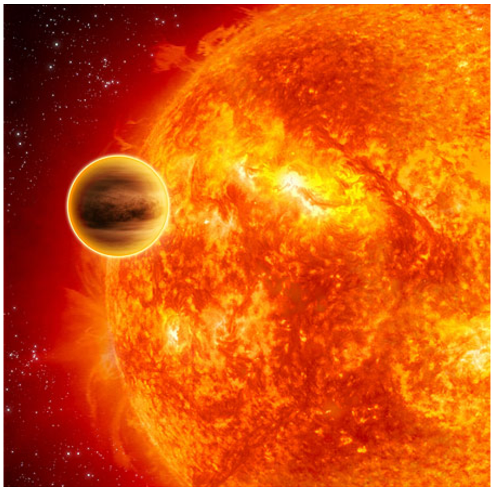
Problem 3 – A very important moment in the history of the universe occurred when the temperature of matter became so low that the electrons and protons in the expanding cosmic plasma cooled enough that stable hydrogen atoms could form. This happened at a temperature of about 4,000 K. How many years after the Big Bang did this occur?

Answer: T = 4000

So t = $1.08x10^{20}$ / $(4000)^2$ = 6.75×10^{12} seconds

Since 1 year = 3.1×10^7 seconds,

T = **220,000 years after the Big Bang**.



Most of the detected exoplanets ae easiest to find when they orbit close to their star. This makes them very hot worlds that are not likely to support life.

The discovery of planets orbiting nearby stars has led to astronomers discovering over 425 planets using a variety of techniques and technologies. Although the majority of these worlds are as large, or larger, than Jupiter, smaller 'super-Earths' are now being detected, and some of these may have the conditions necessary for life.

The planet Gliese-581c orbits a small star located about 20 light years from our sun in the constellation Libra, and takes only 12 days to orbit once around its star.

Problem 1 – Astronomers have measured the mass of Gliese-581c to be about 5.4 times that of our Earth. If the mass of a spherical planet is given by the formula:

$$M = \frac{4}{3}\pi D R^3$$

where R is the radius of the planet and D is the average density of the material in the planet, solve this equation for R in terms of R and D.

Problem 2 - What is the radius of the planet, in kilometers, if the density is similar to solid rock with D = 5500 kg/m^3 and the mass of Earth is 6 x 10^{24} kg?

Problem 3 - What is the radius of the planet, in kilometers, if the density is similar to that of the Ice World Neptune with D = 1600 kg/m^3?

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where R is the radius of the planet and D is the average density of the material in the planet, solve this equation for R in terms of R and D.

Answer:

$R = (3 M / 4\pi D)^{1/3}$

Problem 2 - What is the radius of the planet if the density is similar to solid rock with D = 5500 kg/m^3 and the mass of Earth is 6×10^{24} kg?

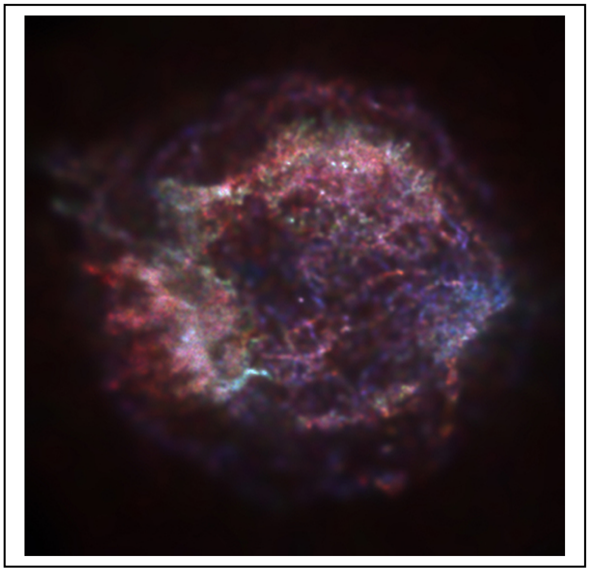
Answer:

$R = (3 \times 5.4 \times 6\times10^{24} / (4 (3.14) 5500))^{1/3} = 1.1 \times 10^7$ meters = **11,000 km**.

Problem 3 - What is the radius of the planet if the density is similar to that of the ice world Neptune with D = 1600 kg/m^3?

Answer:

$R = (3 \times 5.4 \times 6\times10^{24}/(4 (3.14) 1600))^{1/3} = 1.66 \times 10^7$ meters = **16,600 km**.



Once a star explodes as a supernova, the expanding shell of debris expands outwards at speeds of 10,000 km/s to form a growing shell of gas, which can be seen long after the explosion occurred.

The image to the left shows the Cassiopeia-A supernova remnant as revealed by the Chandra X-ray Observatory.

A simple equation approximates the radius of the shell, R, in meters, given the density of the gas it is traveling through, N, in atoms/meter^3, and the total energy, E, of the explosion in Joules.

$$R(E,N,t) = 2.4x10^{8}\left(\frac{E}{N}\right)^{\frac{1}{5}} t^{\frac{2}{5}} \quad \text{meters}$$

Problem 1 – Astronomers can typically determine the size of a supernova remnant and estimate the density and energy, but would like to know the age of the expanding shell. What is the inverse function t(R,E,N) given the above formula?

Problem 2 – From historical data, astronomers might know the age of the supernova remnant, but would like to determine how much energy was involved in creating it. What is the inverse function E(R,N,t)?

Problem 3 – The Cassiopeia-A supernova remnant has an age of about 500 years and a diameter of 10 light years. If 1 light year equals 9.3x10^{12} km, and the average density of the interstellar medium is 10^{6} atoms/meter^{3}, what is was the energy involved in the supernova explosion?

Problem 1 – Astronomers can typically determine the size of a supernova remnant and estimate the density and energy, but would like to know the age of the expanding shell. What is the inverse function t(R,N,E) given the above formula?

Answer: $t(R,N,E) = 1.1x10^{-21}\left(\frac{N}{E}\right)^{\frac{1}{2}} R^{\frac{5}{2}}$ years

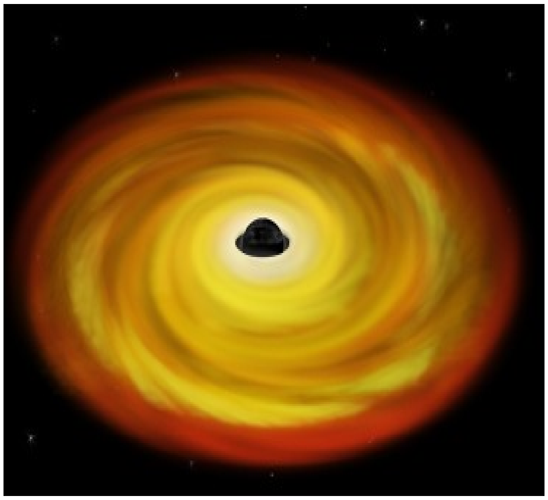
Problem 2 – From historical data, astronomers might know the age of the supernova remnant, but would like to determine how much energy was involved in creating it. What is the inverse function E(R,N,t)?

Answer: $E(R,N,t) = 1.3x10^{-42}\frac{R^5 N}{t^2}$ Joules

Problem 3 – The Cassiopeia-A supernova remnant has an age of about 500 years and a diameter of 10 light years. If 1 light year equals $9.3x10^{12}$ km, and the average density of the interstellar medium is 10^6 atoms/meter^3, what is was the energy involved in the supernova explosion?

Answer: R = 5 x 9.3e^{15} = 4.7x10^{16} m.

E = 1.3x10^{-42} (4.7x10^{16})^5 (10^6)(500)$^{-2}$ Joules
= 1.3x10^{-42} (2.3x10^{83}) (10^6)(4x10^{-6}) Joules
= **1.2x10^{42} Joules**



An artistic rendition of matter flowing into a black hole at the center of an accretion disk.
(Courtesy M.Weiss NASA/Chandra)

Black holes are among the most peculiar objects in our universe. Although they can be detected at great distances, future travelers daring to orbit one of them will experience very peculiar changes. Let's have a look at a small black hole with the mass of our Sun. Its radius will be defined by its horizon size, which is at a distance from the center of the black hole of R=2.8 kilometers.

A distant observer on Earth will watch the clock carried by the Traveler begin to slow down according to the formula:

$$T = \frac{t}{\sqrt{1 - \frac{2.8}{r}}}$$

where t is the time passing on the Traveler's clock, and T being the time interval a distant Observer witnesses.

Problem 1 – The Observer knows that the Traveler's clock is ticking once every second so that t = 1.0. Find the inverse function R(T) that gives the distance of the Traveler, R, from the center of the black hole in terms of the time interval, T, measured by the Observer back on Earth.

Problem 2 – The Observer watches as the Traveler's clock ticks slower and slower. If the Observer measures the ticks at the intervals of T= 5 seconds, 20 seconds and 60 seconds, how close to the event horizon (R=2.8 km) of the black hole, in meters, is the Traveler in each instance?

Problem 1 – The Observer knows that the Traveler's clock is ticking once every second so that T0 = 1.0. Find the inverse function R(T) that gives the distance of the Traveler, R, from the center of the black hole in terms of the time interval, T, measured by the Observer back on Earth.

Answer:

$1 - 2.8/r = (t / T)^2$

$1 - (t / T)^2 = 2.8/r$

$R(T) = 2.8/(1 - (t / T)^2)$ Since t = 1.0

$$R(T) = \frac{2.8}{\left(1 - \frac{1}{T^2}\right)}$$

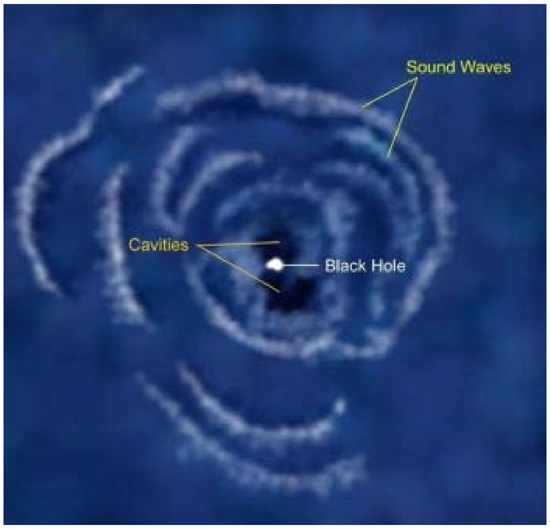
Problem 2 – The Observer watches as the Traveler's clock ticks slower and slower. If the Observer measures the ticks at the intervals of T= 5 seconds, 20 seconds and 60 seconds, how close to the event horizon of the black hole, in meters, is the Traveler in each instance?

Answer: Solve for R which gives the distance to the center of the black hole, and subtract 2.8 km, then multiply by 1000 to get the distance in meters to the event horizon:

For T = 5 seconds; R(5) = 2.8/(1 – 0.04) = 2.92 km from the center,
or 2.92 km – 2.8 km = 0.12 km = **120 meters to the horizon**.

T = 20 seconds R(20) = 2.8/(1-0.0025) = 2.807 km from the center
or 2.807 km – 2.8.000 km = 0.007 km or **7 meters to the horizon**.

T = 60 seconds R(60) = 2.8/(1 – 0.00028) = 2.8008 km from the center
or 2.8008 km – 2.8 km = 0.0008 km or **0.8 meters from the horizon**.



This sketch, based on data from the Chandra Observatory identifies sound waves escaping from the vicinity of the massive back hole in the center of the Perseus galaxy cluster. It takes 20 million years for one wave to move 1 million light years! (Courtesy NASA/Chandra)

The speed of sound in a gas is one of those basic properties that we take for granted, except when we are listening for sirens from fire trucks, trying to find out how far away lightning struck, or when we are playing with helium balloons to sound like Donald Duck at birthday parties.

The speed of sound, S in meters/second, can be calculated from the formula:

$$S = 108\sqrt{\frac{T}{m}}$$

where m is the average molecular mass of the gas in grams/mole, and T is the temperature of the gas in Kelvin degrees.

Problem 1 – What is the inverse function that gives the temperature of the gas in terms of its sound speed T(S)?

Problem 2 – What is the inverse function that gives the composition of the gas, m, in terms of its sound speed m(S)?

Problem 3 - At a temperature of 300 K, the speed of sound is measured to be 450 meters/sec. What is the inferred average molecular mass of the gas?

Problem 1 – What is the inverse function that gives the temperature of the gas in terms of its sound speed T(S)?

Answer:

$$T(S) = \frac{S^2 m}{1164} \quad \text{or} \quad T(S) = 0.00086\, S^2 m$$

Problem 2 – What is the inverse function that gives the composition of the gas, m, in terms of its sound speed m(s)?

Answer:

$$M(S) = 1164 \frac{T}{S^2}$$

Problem 3 - At a temperature of 300 K, the speed of sound is measured to be 450 meters/sec. What is the inferred average molecular mass of the gas?

Answer:

$M = 11664\ (300)/(450)^2$

= **17.3 grams/mole**.

Bok globules in the star forming region IC-2944 photographed by the Hubble Space Telescope.

Stars are formed when portions of interstellar clouds collapse upon themselves and reach densities high enough for thermonuclear fusion to begin to stabilize the cloud against further gravitational collapse.

An important criterion that determined whether a cloud will become unstable and collapse in this way is called the Jeans Criterion, and is given by the formula:

$$M = 2.5x10^{30}\sqrt{\frac{T^3}{N}} \quad \text{kg}$$

where N is the density of the gas in atoms/meter^3, and T is the temperature in degrees Kelvin.

Problem 1 – What is the inverse function that gives the critical density of the gas, N, in terms of its mass and temperature?

Problem 2 – What is the inverse function that gives the critical temperature of the gas for a given density and total mass?

Problem 3 - An astronomer measures an interstellar gas cloud and find it has a temperature of T = 40 K, and a density of N = 10,000 atoms/meter^3. If the observed mass is 200 times the mass of the sun, is this cloud stable or unstable? (1 solar mass = 2 x 10^{30} kilograms)

Problem 1 – What is the inverse function that gives the critical density of the gas, N, in terms of its mass and temperature?

Answer: $N = 6.3x10^{60}\frac{T^3}{M^2}$

Problem 2 – What is the inverse function that gives the critical temperature of the gas for a given density and total mass?

$$T = 5.7x10^{-21}N^{\frac{1}{3}}M^{\frac{2}{3}}$$

Problem 3 - An astronomer measures an interstellar gas cloud and find it has a temperature of T = 40 K, and a density of N = 10,000 atoms/meter^3. If the observed mass is 200 times the mass of the sun, is this cloud stable or unstable? (1 solar mass = 2 x 10^{30} kilograms).

Answer: Use the original equation for M and determine the critical mass for this cloud. If the critical mass is above this, then the cloud will collapse. If it is below this then the cloud is stable.

$$M = 2.5x10^{30}\sqrt{\frac{T^3}{N}} \text{ kg}$$

$$M = 2.5x10^{30}\sqrt{\frac{40^3}{10000}}$$

so M = 6.3 x 10^{30} kilograms. Since 1 solar mass = 2 x 10^{30} kg, M = 3.1 times the mass of the sun.

But the mass of this interstellar cloud is 200 solar masses so since 200 > 3.1 **the cloud must be unstable.**

Graphing Square and Cube-root Functions

Gravity can cause time to run slower than it normally would in its absence. This effect is particularly strong near black holes or neutron stars, which are astronomical objects with very intense gravitational fields. The equation that accounts for gravitational time delays is:

$$T(x) = 4.0 + 1.0\sqrt{1-x}$$

where x is the strength of the gravitational field where the signal is sent.

Problem 1 - What are the domain and range of T(x)?

Problem 2 – How would you obtain the graph of T(x) from the graph of $g(x) = x^{1/2}$?

Problem 3 – Graph the function T(x).

Problem 4 – Evaluate T(x) for the case of a clock located on the surface of a neutron star that has the mass of our sun, a diameter of 20 kilometers, and for which x = 0.20.

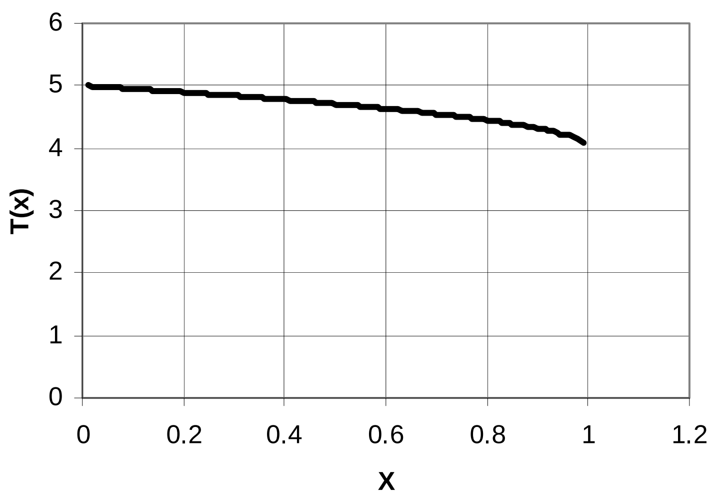
Problem 5 – Evaluate T(x) for the case of a clock located on the surface of a white dwarf star that has the mass of our sun, a diameter of 10,000 kilometers, and for which x = 0.001.

Problem 1 - What are the domain and range of T(x)?

Answer: The domain extends from [-infinity, 1.0], however it should be noted that gravitational fields cannot be negative so the actual physical domain is [0,1.0]
The range extends from [0.0,+5.0]

Problem 2 – How would you obtain the graph of T(x) from the graph of $g(x) = x^{1/2}$?
Answer: Shift $x^{1/2}$ upwards by +4.0, and to the right by +1.0

Problem 3 – Graph the function T(x).



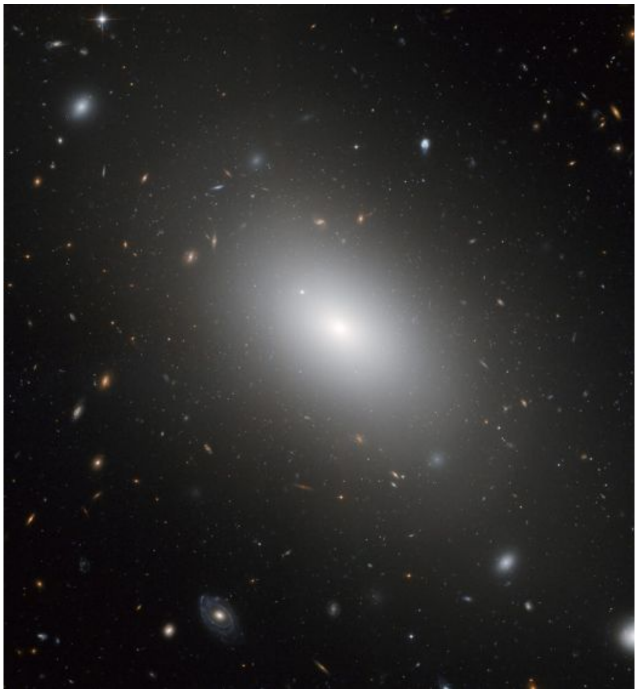
Problem 4 – Evaluate T(x) for the case of a clock located on the surface of a neutron star that has a mass of the sun, a diameter of 20 kilometers, and for which x = 0.20.

Answer: $T(0.20) = 4.0 + (1 – 0.2)^{1/2} = 4.89$

Problem 5 – Evaluate T(x) for the case of a clock located on the surface of a white dwarf star that has a mass of the sun, a diameter of 10,000 kilometers, and for which x = 0.001.

$T(0.001) = 4.0 + (1-0.001)^{1/2} = 4.999$

Solving radical equations



Elliptical galaxies have very simple shapes. Astronomers have measured how the brightness of the galaxy at different distances from its core region obeys a formula similar to:

$$L(r) = \frac{L_0}{(1+ar)^3}$$

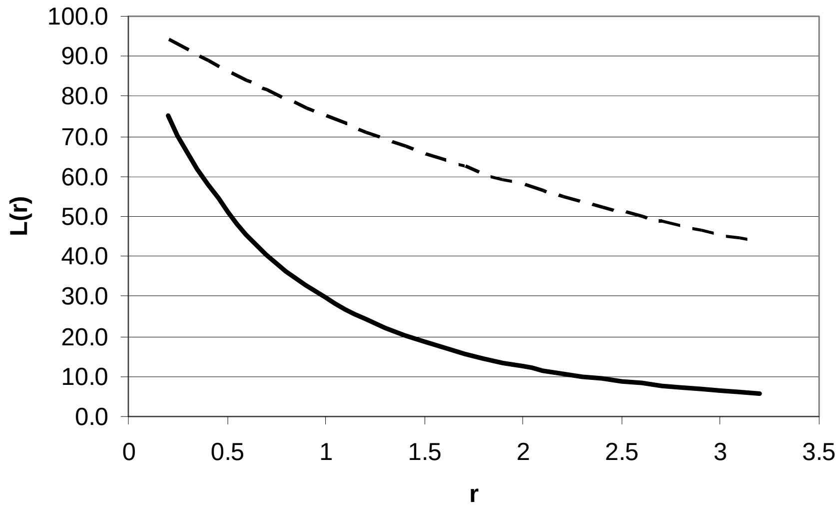
called the de Vaucouleur's Law. In this equation, $\mathbf{L_0}$ is the brightness at the center of the galaxy, **r** is the distance from the center, and **a** is a scaling constant to represent many possible shapes of the same basic form.

Problem 1 – Graph L(r) for the two cases where L_0 = 100 and **a** = 0.5 and **a** = 0.1.

Problem 2 – The above photo is of the elliptical galaxy NGC 1311 obtained by the Hubble Space Telescope. Describe how L(r) relates to what you see in the image of this galaxy?

Problem 3 - An astronomer maps the brightness of an elliptical galaxy and determines that L_0 = 175.76 and that at a distance of r = 5 the brightness of the galaxy has dimmed to 1/100 of L_0. What is the value for the variable **a** that matches this data?

Problem 1 – Graph L(r) for the two cases where L0 = 100 and **a** = 0.5 and **a** = 0.1.



By plotting more curves, students can see what effect changing the shape parameter 'a' has on the modeled brightness profile of a galaxy. This creates a family of functions L(r) for modeling many different types of elliptical galaxies.

Problem 2 – The photo is of the elliptical galaxy NGC 1311 obtained by the Hubble Space Telescope. Describe how L(r) relates to what you see in the image of this galaxy?

Answer: Near r=0 the brightness of the galaxy becomes very intense and the picture shows a bright spot of high intensity. As you move farther from the nucleus of the galaxy, the brightness of the stars fades and the galaxy becomes dimmer at larger distances.

Problem 3 - An astronomer maps the brightness of an elliptical galaxy and determines that L0 = 175.76 and that at a distance of r = 5 the brightness of the galaxy has dimmed to 1/100 of L0. What is the value for the variable **a** that matches this data?

Answer: For L0=175.76, L = 1/100, and r = 5,

$1/100 = 175.76/(5a + 1)^3$

$(5a + 1)^3 = 17576$

$5a + 1 = (17576)^{1/3}$

$5a = 26 – 1$

$a = 5$ so the function that models the brightness of this elliptical galaxy is

$$L(r) = \frac{175.76}{(5r+1)^3}$$

As the universe expanded and cooled, the gases began to form clumps under their own gravitational attraction. As the temperature of the gas continued to cool, it became less able to resist the local forces of gravity, and so larger and larger clouds began to form in the universe. A formula that models the growth of these clouds with temperature is given by the Jeans Mass equation

$$M(T) = \frac{9.0x10^{18}}{(1+9.2x10^{-5}T)^3}$$

where T is the temperature of the gas in Kelvin degrees, and M is the mass of the gas cloud in units of the sun's mass.

Problem 1 – The mass of the Milky Way galaxy is about $3.0x10^{11}$ solar masses, at about what temperature could gas clouds of this mass begin to form?

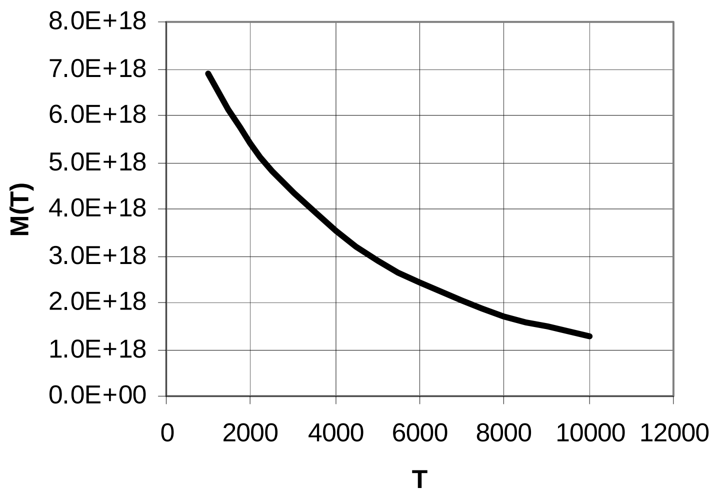
Problem 2 – Graph this function in the range from T= 1,000 K to T = 10,000 K. What was the size of the largest collection of matter that could form at a temperature of 4,000 K?

Problem 1 – The mass of the Milky Way galaxy is about 3.0×10^{11} Msun, at about what temperature could gas clouds of this mass begin to form?

$3 \times 10^{11} = 9 \times 10^{18} / (1 + 9.2 \times 10^{-5}\, T)^3$
$(1 + 9.2 \times 10^{-5}\, T)^3 = 9 \times 10^{18} / 3 \times 10^{11}$
$1 + 9.2 \times 10^{-5}\, T = (3.0\times10^{7})^{1/3}$
$T = (309 - 1)/9.2\times10^{-5}$

T = 3.3 million degrees.

Problem 2 – Graph this function in the range from T= 1,000 K to T = 10,000 K. What was the size of the largest collection of matter that could form at a temperature of 4,000 K?



For T = 4,000 K the maximum mass was 3.5×10^{18} Msun. Note this equals a collection of matter equal to about 12 million galaxies, each with the mass of our Milky Way

Statistics and statistical graphs

The Hydra Galaxy Cluster located 158 million light years from Earth contains 157 galaxies, some as large as the Milky Way.

Astronomers determine whether a galaxy is a member of a cluster by comparing its speed with the average speed of the galaxies in the cluster. Galaxies whose speeds are more than 3 standard deviations from the mean are probably not members.

Galaxy	Speed (km/s)
NGC 3285	3329
NGC 3285b	3149
NGC 3307	3897
NGC 3311	3856
NGC 3316	4033
ESO501G05	4027
ESO436G34	3614
ESO501G13	3504
ESO501G20	4306
ESO437G04	3257
ESO501G40	3686
ESO437G11	4745
ESO501G56	3456
ESO501G59	2385
ESO437G21	3953
ESO501G65	4378
ESO437G27	3867
ESO501G66	3142
ESO501G70	3632
ESO437G45	3786

The average speed of a collection of galaxies is given by

$$s = \frac{1}{N}\sum_{i=1}^{N} v_i$$

and the standard deviation of the speed is given by

$$\sigma = \sqrt{\frac{\sum_{i=1}^{N}(v_i - s)^2}{N-1}}$$

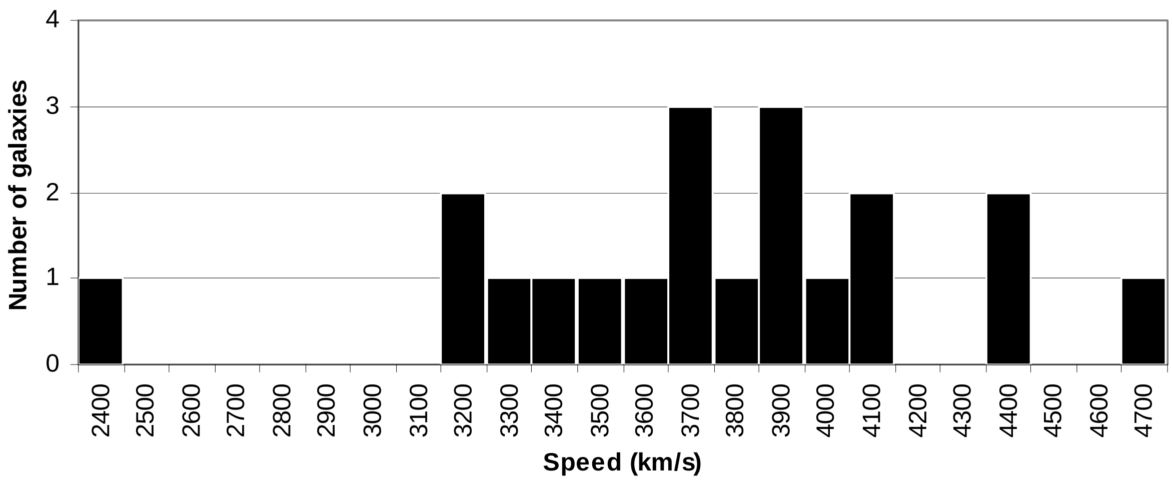
Problem 1 – Create a frequency 'bar' graph of the number of galaxies in 100 km/sec bins between 2400 and 4700 km/sec.

Problem 2 – From the table, what is the average speed, s, of the galaxies in the Hydra I cluster?

Problem 3 – From the table, what is the standard deviation, σ, of the speeds in the table?

Problem 4 - Which galaxies may not be a member of this cluster?

Problem 1 – Create a frequency 'bar' graph of the number of galaxies in 100 km/sec bins between 2400 and 4700 km/sec.



Problem 2 – From the table, what is the average speed of the galaxies in the Hydra I cluster ?
Answer: The sum of the 20 speeds is 74,002 so the average speed
V = 74002/20
V = 3,700 km/sec.

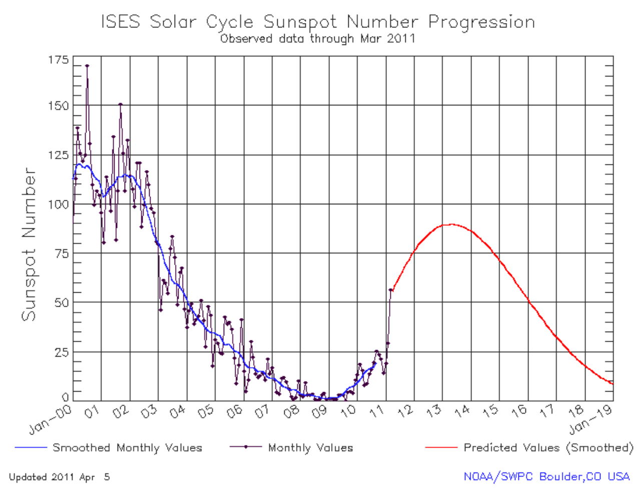
Problem 3 – From the table, what is the standard deviation of the speeds in the table?
Answer:
σ = 504 km/sec

Problem 4 - Which galaxies may not be a member of this cluster?

Answer: From the 3-sigma test, we should probably not include galaxies shows tabulated speeds are greater than s + 3σ and s - 3σ. Since σ = 504 km/s, the speed range is 3,700 + 1,512 = 5,212 km/s and 3700 - 1,512 = 2,188 km/s. Since the tabulated galaxies all have speeds between 2,385 and 4,745 km/s they are probably all members of the cluster.

Data table obtained from Richter, O.-G., Huchtmeier, W. K., & Materne, J. , Astronomy and Astrophysics V111, p195 Table 1 "The Hydra I Cluster of Galaxies'

Statistics and statistical graphs



http://www.swpc.noaa.gov/SolarCycle/

The number of sunspots you can see on the sun varies during an 11-year cycle. Astronomers are interested in both the minimum number and maximum sunspot number (SSN) recorded during the 300 years that they have been observed. Use the following formulae to answer the questions below.

$$s = \frac{1}{N}\sum_{i=1}^{N} v_i \qquad \sigma = \sqrt{\frac{\sum_{i=1}^{N}(v_i - s)^2}{N-1}}$$

Cycle	Minimum	Maximum
1	10	86
2	11	106
3	7	154
4	10	132
5	4	47
6	0	46
7	2	71
8	9	138
9	11	125
10	4	96
11	7	139
12	3	64
13	6	85
14	3	63
15	1	104
16	6	78
17	6	114
18	10	152
19	4	190
20	10	106
21	13	155
22	13	158
23	9	120

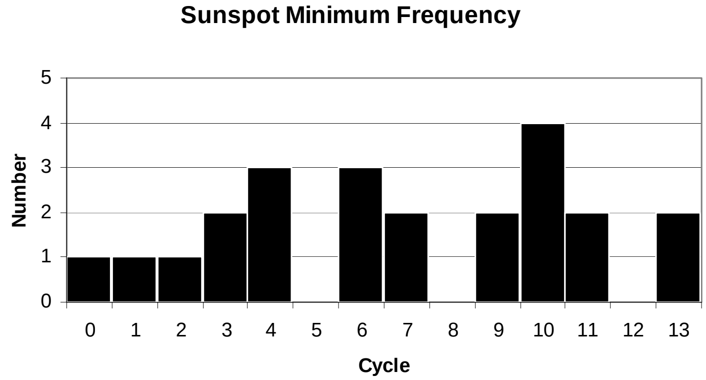
Problem 1 – On two separate graphs, create a frequency histogram of the sunspot A) the SSN minima and B) the SSN maxima with binning of 10 and a range of 0 < SSN < 200

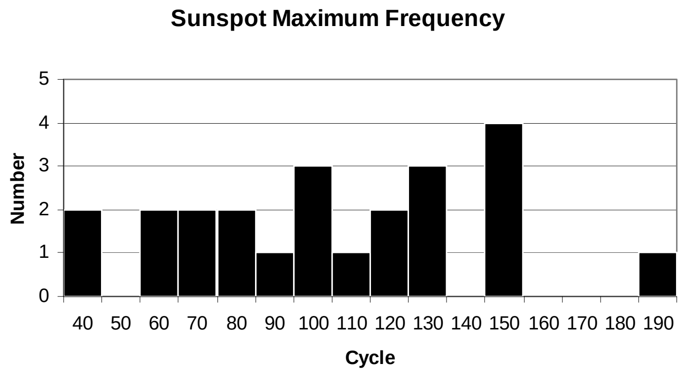
Problem 2 – From the table, and rounded to nearest integer what is A) the average sunspot minimum and maximum? B) The median and mode from the graph?

Problem 3 – From the table, what is the standard deviation, σ, of the minimum and maximum sunspot numbers to the nearest integer?

Problem 4 – Which sunspot cycles appear to be more than 1 standard deviation from the mean value for sunspot maximum?

Problem 1 – On two separate graphs, create a frequency 'bar' graph of the sunspot minima and maxima with binning of 1 for the minimum frequency and 10 for the maximum frequency.





Problem 2 – From the table, what is A) the average 'rounded to nearest integer' sunspot minimum and maximum? B) The median and mode from the graph?

Answer: Minimum: **Average = 7, Median = 7 Mode = 10**
Maximum: Average = 110, Median = 100 Mode = 150

Problem 3 – From the table, what is the standard deviation of the minimum and maximum sunspot numbers to the nearest integer?

Answer: Minimum: $= (320/22)^{1/2} =$ 3.8 or **4**
Maximum: $= (33279/22)^{1/2} =$ 38.9 or **39**

Problem 4 – Which sunspot cycles appear to be more than 1 standard deviation from the mean value for sunspot maximum? (Example: Cycle 19: (190-110)/39 = 2.1 sigma
Answer: **Cycle 3: (1.2); Cycle 5: (-1.7); Cycle 6: (-1.7); Cycle 12: (-1.2); Cycle 14: (-1.2); Cycle 19: (2.1); Cycle 21: (1.2) and Cycle 22: (1.3**).

Data table obtained from the National Geophysical Data Center (NOAA)
ftp://ftp.ngdc.noaa.gov/STP/SOLAR_DATA/SUNSPOT _NUMBERS/YEARLY

A small portion of NASA's WISE star field image (left) shows many faint stars, and dark spaces between them. Because these are digital images, the 'dark' regions are actually defined by measurements of the intensity of the 'empty sky' which can include light from Earth's atmosphere, scattered sunlight, and the digital camera's own electronic 'noise'

Astronomers can 'clean' their images of these contaminating backgrounds by performing simple statistics on the data.

Astronomers used the 'raw' data and isolated a blank region of the image far from any obvious star images. They measured the following intensities for each of 25 pixels in a small square patch of 5 x 5 pixels within the image:

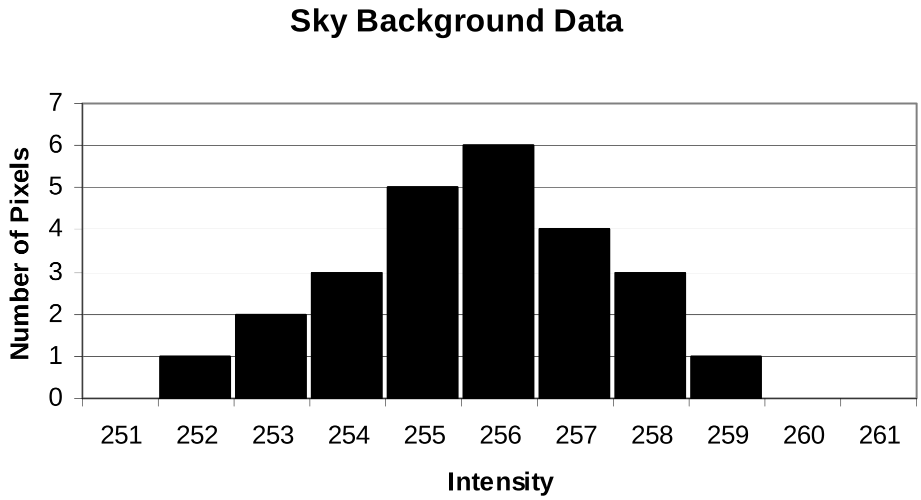
254, 257, 252, 256, 258,
255, 254, 257, 256, 255,
259, 256, 253, 257, 256,
255, 256, 254, 258, 255,
256, 257, 253, 258, 255

Problem 1 – What is the frequency distribution of the background data?

Problem 2 – The average level of the background 'sky' intensity is found by computing the average of the pixel intensities in the dark area of the image. What is the average background intensity, B, of the dark region of the image?

Problem 3 – A measure of the combined instrument 'noise' and sky background variations of the image is found by calculating the standard deviation of the background pixel intensities. What is standard deviation of this patch of the 'dark' sky in the image?

Problem 1 – What is the frequency distribution of the background data?



Problem 2 – The average level of the background 'sky' intensity is found by computing the average of the pixel intensities in the dark area of the image. What is A) the average background intensity, B, of the dark region of the image? B) The median intensity? C) The mode intensity?

Answer: **B = 6392/25 = 255.7**
Median = 256
Mode = 256

Problem 3 – A measure of the combined instrument 'noise' and sky background variations of the image is fund by calculating the standard deviation of the background pixel intensities. What is standard deviation of this patch of the 'dark' sky in the image?

Answer: sigma = $(73.44/25)^{1/2}$
sigma = 1.7