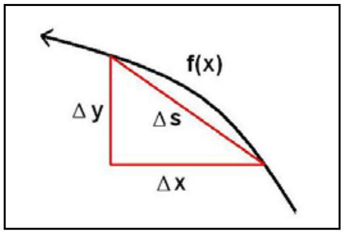
Calculating Arc Lengths of Simple Functions- I



Spirals are found in many different places in astronomy, from the shape of the arms in a 'spiral' galaxy, to the trajectory of a spacecraft traveling outward from Earth's orbit at constant velocity. Figuring out spiral lengths requires a bit of calculus. Here's how it's done:

$$\Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

Step 1: Study the figure above, and use the Pythagorean Theorem to determine the hypotenuse length in terms of the other two sides. It should look like the equation to the left.

$$\Delta s = \sqrt{1 + \left(\frac{\Delta y}{\Delta x}\right)^2}\ \Delta x$$

Step 2: Factor out the **Δx** to get a new formula.

$$S = \int_A^B \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\ dx$$

Step 3: Following the basic techniques of calculus, 'take the limit' and allow the deltas to become differentials, then use the integral calculus to sum-up all of the differentials along the curve defined by y = F(x), and between points A and B, to get the fundamental arc-length formula.

$$S = \int \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\ d\theta$$

The arc length formula can be re-written in polar coordinates too. In this case, the function, y = F(x) has been replaced by the polar function **r(θ).**

Problem 1) Find the arclength for the line y = mx + b from x=3 to x=10.

Problem 2) Find the arclength for the parabolic arc defined by $y = x^2$ from x=1 to x=5.

Problem 3) Find the arclength for the logarithmic spiral $R(\theta) = e^{b\theta}$ from $\theta = 0$ to $\theta = 4\pi$ if b = 1/2.

Problem 4) The spiral track on a CDROM is defined by the simple formula $R = k\theta/2\pi$, where k represents the width of each track of data. If k = 1.5 microns, how long is the spiral track, in meters, for a standard r =6.0-cm disk if only the radius between 2.5cm and 5.8 cm is used? (use π = 3.14159)

Answer Key

Problem 1) dy/dx = m, so the integrand becomes $(1 + m^2)^{1/2}$ dx. Because m is a constant independent of x, the integral is just $(1 + m^2)^{1/2}$ (10 - 3) = **$7 (1+ m^2)^{1/2}$.**

Problem 2) dy/dx = 2x and the integrand becomes $(1 + 4x^2)^{1/2}$ dx. This can be integrated by using the substitution 2x = sinh(u) , and dx = (1/2)cosh(u) du, so that the integrand becomes 1/2 $\cosh^2(u)$ du. This is a fundamental integral with the solution **1/2 [sinh(2u)/4 + u/2 + C]**.

Limits: The limits go from x=1 to x=5, but since x = 1/2 sinh(u), the limits re-expressed in terms of u become $u_1 = \sinh^{-1}(2)$ = 1.44 and $u_2 = \sinh^{-1}(10)$ = 3.00 so evaluating the definite integral leads to 1/2 (sinh(6.00)/4 + 3.00/2) - 1/2 (sinh(2.88)/4 + 1.44/2) = 25.96 - 1.47 = 24.49. Because the limits are provided to three significant figures, the answer to the same number of significant figures will be 24.5

Problem 3) We use the polar form of the arclength formula. First perform the differentiation of r(θ) to get dr/dθ = b $e^{b\theta}$. Then after substitution, the integrand becomes $(e^{2b\theta} + b^2e^{2b\theta})^{1/2}$ dθ which after simplification then becomes $e^{b\theta}(1 + b^2)^{1/2}$ dθ . This is easily integrated to get (1/b) $(1 + b^2)^{1/2}$ $e^{b\theta}$ + **C**. Since b = 1/2, we get the simpler form 2.24 $e^{\theta/2}$ + C. This can be evaluated between the two limits for θ to get 2.24 (534.86 - 1) = 1,195.85. Because the limits are provided to three significant figures, the answer to the same number of significant figures will be 1,200.

Problem 4) The radial function is designed so that every 2π radians, the radius advances by k units in length. Because r = kθ/2π, the integrand becomes $((k/2\pi)^2\theta^2 + (k/2\pi)^2)^{1/2}$ dθ or $(k/2\pi)(1 + \theta^2)^{1/2}$ dθ. This can be simplified using the hyperbolic trig identity $1 + \sinh^2(x) = \cosh^2(x)$ where we have used the substitution θ = sinh(x). This also means that dθ = cosh(x) dx. Then the integrand becomes $\cosh^2(x)$ dx. The integral is then a fundamental integral with the solution (**k/2π) [sinh(2x)/4 + x/2 + C] which as for all indefinite integrals includes the constant of integration C**.

Limits: How many radians does the spiral take up? Lower integral limit: θ = 2 π x 2.5 cm/1.5 microns = 33,333 π. Upper integral limit: θ = 2 π x 5.8 cm/1.5 microns = 77,333 π. But θ = sinh(x) so the limits become x(lower) = $\sinh^{-1}$(33,333 π) = 12.252 to x(upper) = $\sinh^{-1}$(77,333 π) = 13.094.

The definite integral is then (1.5 microns/2π) x [(sinh(26.188)/4 +13.094/2 + C) - (sinh(24.504)/4 + 12.252/2 + C) = (0.0000015 meters/2π) (2.95 - 0.548) x 10^{10} = 5734 meters or 5.734 kilometers. Because the problem only gives 1.5 and 6.0 to two significant figures, this becomes the maximum accuracy of the available numbers, so the answer to the same number of significant figures will be **5.7 kilometers!**